Initiation of cracks in Griffith's theory: an argument of continuity in favor of global minimization

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Abstract The initiation of a crack in a sound body is a real issue in the setting of Griffith's theory of brittle fracture. If one uses the concept of critical energy release rate (Griffith's criterion), it is in general impossible to initiate a crack. On the other hand, if we replace it by a least energy principle (Francfort-Marigo's criterion), it becomes possible to predict the onset of cracking in any circumstance. However this latter criterion can appear too strong. We propose here to reinforce its interest by an argument of continuity. Specifically, we consider the issue of the initiation of a crack at a notch whose angle ω is considered as a parameter. The result predicted by the Griffith criterion is not continuous with respect to ω, since no initiation occurs when $\omega > 0$ while a crack initiates when $\omega = 0$. In contrast, the Francfort-Marigo's criterion delivers a response which is continuous with respect to ω, even though the onset of cracking is necessarily brutal when $\omega > 0$. The theoretical analysis is illustrated by numerical computations.

Keywords Fracture · Stability · Energy minimization · Variational methods

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1 Introduction.

Griffith's theory of fracture [22] remains the most used in Engineering, [5], [9], [27], [28]. Its main advantage is its simplicity in terms of material behavior, because it only requires the identification of the two Lamé coefficients λ, μ and the surface energy density G_c for an isotropic brittle material. However, there exist several ways to set the problem of crack propagation while staying within the framework of Griffith's assumptions. (This lack of uniqueness is in fact the mark that none of those ways is perfect.) We are interested here in two of them. The first one, called in this paper the *G-law*, which is also the most used, is the law based on the concept of critical energy release rate requiring that a crack can propagate only when the potential energy release rate $\mathcal{G}$ is equal to G_c. One of the drawbacks of the energy release rate criterion is its incapacity to account for crack initiation in a body which does not contain a preexisting crack. That leads Francfort and Marigo in [19] to reformulate the law in terms of minimization of the total energy of the body. This revisited Griffith energy principle, the so-called *FM-law*, is equivalent to the critical energy release rate in a certain number of cases, as it is recalled in this paper, but is (in general) quite different as far as the crack initiation is concerned. In particular, with the least energy principle, it becomes possible to predict the onset of cracking in a sound body. However, the price to pay is that the onset of cracking is necessarily brutal in the sense that a crack of finite length appears at a critical load. The reason is that the elastic response (without any crack) is always a (local) minimum of the energy. Therefore the body has to jump from a local minimum to another (local or global) minimum. This revisited Griffith theory, which simply consists in formalizing the seminal Griffith idea, provided the adequate mathematical framework to obtain new results by inserting fracture mechanics into a modern variational approach, [11], [18], [10]. In return, several criticisms can be made against this principle of least energy when it is applied to predict the crack initiation. One of them is that the body must cross over an energy barrier to jump from one well to the other. The presence of that energy barrier (which ensures the stability of the elastic response) is essentially due to the fact that Griffith's theory does not contain a critical stress and allow singular stress fields. Of course it is possible to introduce this concept of critical stress by leaving Griffith's setting. It is the essence of cohesive force models ([35], [12], [13], [26], [8], [17]) in the spirit of Dugdale's and Barenblatt's works, cf. [15], [1] and [4]. But that leads to a complexification of the modeling which can be considered as unnecessary when one is only interested in the introduction of an initiation criterion within Griffith's theory.

In this paper we do not leave Griffith's setting and will continue to compare the two formulations. We will show that the latter, the *FM-law*, based on energy minimization, enjoys the fundamental property of delivering a continuous response with respect to the data whereas the former one, the *G-law*, formulated in terms of the energy release rate, does not. This major difference appears in particular when it is question of crack initiation. This result greatly militates in favor of the minimization principle. Specifically, we consider the case of a two-dimensional body which contains a notch the opening ϵ of which is taken as a parameter. The limit case $\epsilon = 0$ corresponds to an initial crack. Assuming that the crack will appear (or propagate) at the tip of the notch (or of the preexisting crack) and that the crack path is known, the problem consists in determining the evolution $\ell^\epsilon(t)$ of the crack length with the loading parameter t. The evolution depends of course on ϵ and on the chosen criterion of propagation. Since the concept of crack in Continuum Mechanics— where a crack is considered as a surface of discontinuity— is an idealization of the reality, a criterion of initiation or of propagation can be considered as physically acceptable only if it is stable under small perturbations. In other words, the law is acceptable only if it delivers a response which continuously depends on the geometrical or material parameters of the problem. In the present case that means that the initiation and the propagation of a crack from the tip of a notch whose angle is small must be close to those corresponding to the evolution from a preexisting crack. In mathematical terms that means that the function $t \mapsto \ell^\epsilon(t)$ must converge (in a sense to be precised) to $t \mapsto \ell^0(t)$ when ϵ goes to 0. Unfortunately, the critical energy release rate criterion does not enjoy this continuity property. On the contrary, the least energy criterion does.

Let us summarize here the reasons of these differences (they will be developed in the paper). Since the singularity at the tip of a notch ($\epsilon > 0$) is "weak", the energy release rate $\mathcal{G}_\epsilon(t, \ell)$ associated with a crack of small length ℓ (starting from the tip of the notch) goes to 0 when ℓ goes to 0, *i.e.* $\lim_{\ell\to 0} \mathcal{G}_\epsilon(t, \ell) = 0, \forall t$. Consequently, no crack will appear if we use the critical energy release rate criterion, *i.e.* $\ell^\epsilon(t) = 0\ \forall t$. On the other hand, if we consider a preexisting crack ($\epsilon = 0$), then the singularity is strong enough so that $\mathcal{G}_0(t, 0) = \mathsf{G}_0^0 t^2$ with $\mathsf{G}_0^0 > 0$ (in general). Consequently, the critical energy release rate criterion predicts that the crack will propagate at a (finite) critical loading

$\mathsf{t}_\mathsf{i}^0 = \sqrt{\mathsf{G}_c/\mathsf{G}_0^0}$. What happens for $t > \mathsf{t}_\mathsf{i}^0$ depends on the convexity properties of the potential energy as a function of ℓ, but in any case there is no continuity of the response with respect to ϵ at $\epsilon = 0$. In contrast, we will show that this continuity property holds if we define the evolution from the least energy criterion. In particular, when $\epsilon > 0$, the least energy criterion predicts that a crack of finite length ℓ_i^ϵ suddenly appears at $t = \mathsf{t}_\mathsf{i}^\epsilon$, then propagates continuously with t. Moreover, we prove that $\lim_{\epsilon\to 0} \ell_\mathsf{i}^\epsilon = 0$ and $\lim_{\epsilon\to 0} \mathsf{t}_\mathsf{i}^\epsilon = \mathsf{t}_\mathsf{i}^0$ and that the height of the energy barrier tends to 0 with ϵ.

Even if the proofs are given in the restricted setting of anti-plane elasticity, the results and conclusions would remain unchanged in plane elasticity. (The proofs become a little more complicated but the numerical computations are essentially the same.) In the same manner, the quasi-static assumption which is adopted throughout the analysis is not essential. Indeed, as far as the initiation of a crack at the tip of a notch is concerned, Griffith's criterion remains unable to predict the initiation in dynamics, because the singularity is of the same type as in statics and hence the energy release rate vanishes also in dynamics.

The paper is organized as follows. In section 2 we introduce the two evolution laws and compare them within a general framework involving only a few basic regularity conditions for the energies. Sections 3 and 4 are devoted to the problem of tearing of a notch-shaped body. In section 3 we check all the regularity conditions of the energy whereas we present in section 4 the numerical results obtained by the finite element method. A short appendix contains a Lemma which is used several times in section 3.

The notation is quite classic. Derivatives with respect to coordinates are denoted with a comma, like $u_{,i}$ for $\partial u/\partial x_i$. $L^2(\Omega)$ stands for the set of square integral functions over Ω (for the Lebesgue measure), $H^m(\Omega)$ with $m = 1, 2, \cdots$ for the usual Sobolev space of functions which are and whose weak partial derivatives up to the order m are in $L^2(\Omega)$. The qualifiers positive (resp. negative) and increasing (resp. decreasing) are equivalent to strictly positive (resp. strictly negative) and strictly increasing (resp. strictly decreasing).

2 General setting of Griffith's theory

We recall the main ingredients of Griffith's theory [22] in a quasi-static two-dimensional setting, formulate the two evolution laws of crack propagation that we will compare throughout the paper and establish the first general properties of these laws. The definitions adopted here for the two laws are slightly different from those given in the previous publications [19,4]. Several results refine the previous ones, the improvement being due to a weakening of the hypotheses on the energies. The interested reader should also refer to other publications devoted to similar comparisons of the so-called Griffith and Francfort-Marigo formulations, e.g. [36], [37].

2.1 The main ingredients

We consider a two-dimensional brittle-elastic body submitted to a proportional loading and in which a crack initiates and propagates along a predefined path. At this stage, no assumption is made for the loading except its proportional character. The loading can consist in given surface forces as well as in body forces or in prescribed displacements of the boundary. In the same way, the only requirement on the behavior of the medium is to be linearly elastic in its sound parts and to have a surface energy density *à la Griffith* along the predefined path. The medium can be heterogeneous or anisotropic. Assuming that the path is a (smooth) curve whose arc-length is ℓ, denoting by $t > 0$ the increasing amplitude of the loading, the problem consists in finding the function $t \mapsto \ell(t)$ giving the evolution of the tip (or equivalently, of the length) of the crack with the loading. For that, we will introduce and analyze two evolution laws, both formulated in energetic terms and in a quasi-static setting.

Due to the fact that the behavior is linearly elastic and that the loading is proportional, the potential energy of the body at equilibrium under the loading t and with a crack of length ℓ can be read as

$$\mathcal{P}(t,\ell) = t^2 \mathsf{P}(\ell). \tag{1}$$

Since the crack path is given and since we adopt the Griffith assumption for the surface energy density, the surface energy of the body only depends on the crack length, say $\mathsf{S}(\ell)$. Therefore, the total energy of the body (at equilibrium under the loading t and with a crack of length ℓ) is given by

$$\mathcal{E}(t,\ell) = t^2 \mathsf{P}(\ell) + \mathsf{S}(\ell). \tag{2}$$

We make the following assumptions on these energies:

Hypothesis 1 *The functions $\ell \mapsto \mathsf{P}(\ell)$ and $\ell \mapsto \mathsf{S}(\ell)$ are continuously differentiable in the interval $[0, L]$. Moreover, P is decreasing, $\mathsf{S}(0) = 0$ and the derivative S' of S is positive.*

Although these hypotheses are rather natural, they have to be checked in each case because the involved functions depend on the different parameters of the problem (geometry, behavior and loading). For example, when the medium is homogeneous and isotropic, then the surface energy simply reads $\mathsf{S}(\ell) = \mathsf{G}_c\ell$, where G_c is the surface energy density. Thus, the hypotheses on S are satisfied. On the other hand, the continuous differentiability of P and S are not always ensured when the medium is heterogeneous or when the crack path is not sufficiently smooth. Let us note that the derivatives are supposed to exist at the ends of the interval, what means that the following limits exist:

$$\mathsf{P}'(0) := \lim_{\ell \downarrow 0} \frac{\mathsf{P}(\ell) - \mathsf{P}(0)}{\ell}, \qquad \mathsf{P}'(L) := \lim_{\ell \uparrow L} \frac{\mathsf{P}(L) - \mathsf{P}(\ell)}{L - \ell}$$

with similar definitions for $\mathsf{S}'(0)$ and $\mathsf{S}'(L)$.

The potential energy release rate of the body when the loading is t and the crack length ℓ is given by

$$\mathcal{G}(t,\ell) = -t^2 \mathsf{P}'(\ell) \tag{3}$$

and is non negative since P is decreasing.

The ratio between the potential energy release rate and the surface energy rate, which characterizes the competition between the two forms of energy, will play a fundamental role in the sequel. That leads to the

Definition 2 *Let $\mathsf{g}(\ell)$ be the potential energy release by surface energy created at $t = 1$ when the crack length is ℓ,*

$$\mathsf{g}(\ell) := -\frac{\mathsf{P}'(\ell)}{\mathsf{S}'(\ell)}, \tag{4}$$

g is a continuous non negative function of ℓ on $[0, L]$.

2.2 The two evolution laws

We are now in position to introduce the two evolution laws. The first one, called the *G-law*, is the usual Griffith law based on the critical potential energy release rate criterion, see [5,30,28]. In essence, this law only investigates smooth (*i.e.* at least continuous) evolutions of the crack length with the loading. It consists in the three following items:

Definition 3 (*G-law*) *Let $\ell_0 \in [0, L]$. A continuous function $t \mapsto \ell(t)$ is said satisfying (or solution of) the* G-law *in the interval $[t_0, t_1]$ with the initial condition $\ell(t_0) = \ell_0$, if the three following properties hold*

1. **Irreversibility:** *$t \mapsto \ell(t)$ is not decreasing;*
2. **Energy release rate criterion:** $\mathcal{G}(t, \ell(t)) \leq \mathsf{S}'(\ell(t)), \quad \forall t \in [t_0, t_1]$*;*
3. **Energy balance:** *$\ell(t)$ is increasing only when $\mathcal{G}(t, \ell(t)) = \mathsf{S}'(\ell(t))$.*

The third item means that if $\mathcal{G}(t, \ell(t)) < \mathsf{S}'(\ell(t))$ at some t, then $\ell(t') = \ell(t)$ for every t' in a certain neighborhood $[t, t+h)$ of t. It implies that the release of potential energy is equal to the created surface energy when the crack propagates. Consequently, if $t \mapsto \ell(t)$ is absolutely continuous, then the third item is equivalent to $\frac{\partial \mathcal{E}}{\partial \ell}(t, \ell(t))\dot{\ell}(t) = 0$ for almost all t and the following equality holds for almost all t:

$$\frac{d}{dt}\mathcal{E}(t, \ell(t)) = \frac{\partial \mathcal{E}}{\partial t}(t, \ell(t)). \tag{5}$$

A major drawback of the *G-law* is to be unable to take into account discontinuous crack evolutions, what renders it void in many situations as we will see in the next subsection. It must be replaced by another law which admits discontinuous solutions. Another motivation of changing the *G-law* is to reinforce the second item by introducing a full stability criterion, see [19],[38], [33], [20], [4]. Specifically, let us consider the following local stability condition

$$\forall t \geq 0, \exists h(t) > 0 \quad : \quad \mathcal{E}(t, \ell(t)) \leq \mathcal{E}(t, l) \quad \forall l \in [\ell(t), \ell(t) + h(t)], \tag{6}$$

which requires that the total energy at t is a “unilateral” local minimum. (The qualifier unilateral is added because the irreversibility condition leads to compare the energy at t with only that corresponding to greater crack length, see [4]). Taking $l = \ell(t) + h$ with $h > 0$ in (6), dividing by h and passing to the limit when $h \to 0$, we recover the critical energy release rate criterion. Thus, the second item can be seen as a first order stability condition, weaker than (6). A stronger requirement consists in replacing local minimality by global minimality. It was the condition introduced by Francfort-Marigo in [19] and that we will adopt here. Thus, the second evolution law, called *FM-law*, consists in the three following items

Definition 4 (*FM-law*) *A function $[0, +\infty) \ni t \mapsto \ell(t) \in [0, L]$ is said satisfying (or solution of) the* FM-law *if the three following properties hold*

1. **Irreversibility:** *$t \mapsto \ell(t)$ is not decreasing;*
2. **Least Energy criterion:** $\mathcal{E}(t, \ell(t)) \leq \mathcal{E}(t, l), \quad \forall t \geq 0$ *and* $\forall l \in [\ell(t), L]$*;*
3. **Energy balance:** $\mathcal{E}(t, \ell(t)) = \int_0^t \frac{\partial \mathcal{E}}{\partial t'}(t', \ell(t'))\, dt', \quad \forall t \geq 0.$

Let us note that the irreversibility condition is unchanged, while the energy balance condition is now written as the integrated form of (5), what does not require that $t \mapsto \ell(t)$ be continuous. Note also that the energy balance implies $\ell(0) = 0$ because $0 = \mathcal{E}(0, \ell(0)) = \mathsf{S}(\ell(0))$, and that the second item is automatically satisfied at $t = 0$ because S is increasing.

2.3 Some elements of comparison

We establish in this subsection several general results for the two evolution laws under the assumptions made in Section 2.1. Some of those results were also obtained in [19, 4] but with more restrictive assumptions. A comparison between *G-law* and *FM-law* is proposed also in [37].

As regards the initiation of cracks with the *G-law* we have

Proposition 1 *If* $\mathsf{P}'(0) = 0$*, then the unique solution of the* G-law*, in* $[0, +\infty)$ *with* $\ell_0 = 0$*, is* $\ell(t) = 0$*,* $\forall t \geq 0$*. If* $\mathsf{P}'(0) < 0$*, then* $\ell(t) = 0$ *satisfies the* G-law *only in the interval* $[0, \mathsf{t_i}]$ *with*

$$\mathsf{t_i} = \mathsf{g}(0)^{-1/2} \tag{7}$$

and is the unique solution in this interval.

Proof. If $\mathsf{P}'(0) = 0$, then $\forall t \geq 0$, $0 = \mathcal{G}(t, 0) < \mathsf{S}'(0)$, hence $\ell(t) = 0$ is a solution. The uniqueness follows from the initial condition and the energy balance. If $\mathsf{P}'(0) < 0$, then $-t^2\mathsf{P}'(0) = \mathcal{G}(t, 0) \leq \mathsf{S}'(0)$ if and only if $t \in [0, \mathsf{t_i}]$. Since the inequality is strict when $t \in [0, \mathsf{t_i})$, then $\ell(t) = 0$ is the unique solution in this interval because of the initial condition and the energy balance. By continuity, it is also the unique solution in the closed interval $[0, \mathsf{t_i}]$. □

Thus, if $\mathsf{P}'(0) = 0$, no initiation of crack is possible with the *G-law*, whereas if $\mathsf{P}'(0) < 0$, a crack should appear at $\mathsf{t_i}$. But what happens for $t > \mathsf{t_i}$ in this latter case depends on convexity properties of the energy by virtue of the following

Proposition 2 *Assume that* $\mathsf{P}'(0) < 0$ *and let* $0 < \ell_\mathsf{f} \leq L$*. The* G-law *admits a solution in the interval* $[\mathsf{t_i}, \mathsf{t_f}]$ *such that the crack length grows from* 0 *to* ℓ_f *if and only if* g *is decreasing in the interval* $[0, \ell_\mathsf{f}]$*. In such a case, the solution is unique and given by*

$$\ell(t) = \mathsf{g}^{-1}(t^{-2}), \qquad \mathsf{t_f} = \mathsf{g}(\ell_\mathsf{f})^{-1/2}. \tag{8}$$

Proof. Let us assume that the *G-law* admits a solution. Let us first prove that $t^2\mathsf{g}(\ell(t)) = 1$ for all $t \in [\mathsf{t_i}, \mathsf{t_f}]$. It is true at $\mathsf{t_i}$ by virtue of (7). Let us assume that it is not true for some t and hence that $t^2\mathsf{g}(\ell(t)) < 1$ because of the second item of the *G-law*. By continuity, the inequality holds in an interval $(t_1, t]$. Taking for t_1 the lowest bound, we have necessarily $t_1^2\mathsf{g}(\ell(t_1)) = 1$ (because $\mathsf{t_i}^2\mathsf{g}(0) = 1$). But, by virtue of the energy balance, we must also have $\ell(t_1) = \ell(t)$ and we obtain a contradiction with $1 = t_1^2\mathsf{g}(\ell(t_1)) > t^2\mathsf{g}(\ell(t_1))$ and $t_1 < t$. Hence $t^2\mathsf{g}(\ell(t)) = 1$ for all $t \in [\mathsf{t_i}, \mathsf{t_f}]$. Let ℓ_1 and ℓ_2 be such that $0 \leq \ell_1 < \ell_2 \leq \ell_\mathsf{f}$. By continuity of $t \mapsto \ell(t)$ and because of the irreversibility condition, there exist t_1 and t_2 with $t_1 < t_2$ such that $\ell(t_1) = \ell_1$ and $\ell(t_2) = \ell_2$. Therefore, the strict monotonicity of g follows from $1 = t_1^2\mathsf{g}(\ell_1) = t_2^2\mathsf{g}(\ell_2)$.

Conversely, let us assume that g is decreasing. By virtue of the first part of the proof, if a solution exists, then it necessarily satisfies $t^2\mathsf{g}(\ell(t)) = 1$ for all $t \in [\mathsf{t_i}, \mathsf{t_f}]$ and hence is given by (8). Since this function satisfies the three items, it is the unique solution of the *G-law*. □

Let us now consider the *FM-law*. We first show that the *FM-law* is equivalent to the minimization problem $\min_{l \in [0,L]} \mathcal{E}(t, l)$.

Proposition 3 *Under the assumptions of Section 2.1, a function* $t \mapsto \ell(t)$ *satisfies the* FM-law *if and only if, at each* t*,* $\ell(t)$ *is a minimizer of* $l \mapsto \mathcal{E}(t, l)$ *over* $[0, L]$*. Therefore, the* FM-law *admits at least one solution and each solution grows from* 0 *to* L*.*

PROOF. The proof is divided into 3 steps.

Step 1 : *The minimization problem admits at least one solution. Each solution is not decreasing with* t*, growing from 0 to* L*.*

Since $l \mapsto \mathcal{E}(t,l)$ is continuous on the compact $[0,L]$, it reaches its greatest lower bound. For $t \geq 0$, let $\ell(t)$ be a minimizer of $l \mapsto \mathcal{E}(t,l)$ over $[0,L]$. Since $\mathcal{E}(0,l) = \mathsf{S}(l)$ and since S is increasing, $\ell(0) = 0$. For $0 \leq t_1 < t_2$, let us show that $\ell(t_1) \leq \ell(t_2)$. Indeed, since $t_1^2\mathsf{P}(\ell(t_1)) + \mathsf{S}(\ell(t_1)) \leq t_1^2\mathsf{P}(\ell(t_2)) + \mathsf{S}(\ell(t_2))$ and $t_2^2\mathsf{P}(\ell(t_1)) + \mathsf{S}(\ell(t_1)) \geq t_2^2\mathsf{P}(\ell(t_2)) + \mathsf{S}(\ell(t_2))$, we get $\mathsf{P}(\ell(t_1)) \geq \mathsf{P}(\ell(t_2))$ and hence $\ell(t_1) \leq \ell(t_2)$ because P is decreasing. Let us now prove that $\lim_{t\to\infty} \ell(t) = L$. Let $\ell_\infty := \lim_{t\to\infty} \ell(t) \leq L$ (the limit exists because $\ell(t)$ is monotone and even L can be reached at a finite t). We have $t^2\mathsf{P}(L) + \mathsf{S}(L) \geq t^2\mathsf{P}(\ell(t)) + \mathsf{S}(\ell(t))$ for all t. Dividing by t^2 and passing to the limit when t goes to ∞, we get $\mathsf{P}(L) \geq \mathsf{P}(\ell_\infty)$ and the result follows because P is decreasing.
⊡

Step 2 : *Any solution of the minimization problem satisfies also the* FM-law.

Let $t \mapsto \ell(t)$ be a solution of the minimizing problem. As we proved in the first step, it satisfies the irreversibility condition. It satisfies by definition the second item of the *FM-law*. It remains to check that it satisfies the energy balance. First, since $\mathcal{E}(t_1,\ell(t_1)) \leq \mathcal{E}(t_1,\ell(t_2)) \leq \mathcal{E}(t_2,\ell(t_2))$ when $t_1 \leq t_2$, $t \mapsto \mathcal{E}(t,\ell(t))$ is monotone and hence differentiable almost everywhere. Since $t \mapsto \ell(t)$ is also monotone, it is differentiable almost everywhere. Consequently, at almost all t, we have

$$\frac{d}{dt}\mathcal{E}(t,\ell(t)) = 2t\mathsf{P}(\ell(t)) + \frac{\partial\mathcal{E}}{\partial\ell}(t,\ell(t))\dot{\ell}(t). \tag{9}$$

Moreover $t \mapsto \mathcal{E}(t,\ell(t))$ is locally lipschitzian (and hence absolutely continuous). Indeed, for all $t \geq 0$ and all $h > 0$ we have

$$\begin{aligned} \mathcal{E}(t+h,\ell(t+h)) - \mathcal{E}(t,\ell(t)) &\leq \mathcal{E}(t+h,\ell(t)) - \mathcal{E}(t,\ell(t)) \\ &= (2ht+h^2)\mathsf{P}(\ell(t)) \\ &\leq (2ht+h^2)\mathsf{P}(0). \end{aligned}$$

Since $\mathcal{E}(t,\ell(t)) \leq \mathcal{E}(t,\ell(t\pm h))$ for $t > 0$ and $h > 0$ sufficiently small, dividing by h and passing to the limit when $h \to 0$, we obtain for almost all t

$$\frac{\partial\mathcal{E}}{\partial\ell}(t,\ell(t))\dot{\ell}(t) = 0.$$

Inserting into (9), the energy balance follows by integration. ⊡

Step 3 : *Any solution of the* FM-law *is a solution of the minimization problem.*

Let $t \mapsto \ell(t)$ be a solution of the *FM-law* (its existence is ensured thanks to the previous steps) and let $\mathsf{E}(t)$ be the minimum of $l \mapsto \mathcal{E}(t,l)$ over $[0,L]$ with $l(t)$ a minimizer. We have to prove that $\mathcal{E}(t,\ell(t)) = \mathsf{E}(t)$ for all t. The third item of the *FM-law* gives $\mathcal{E}(0,\ell(0)) = \mathsf{E}(0) = 0$ and $\ell(0) = 0 = l(0)$. Let us assume that $\mathcal{E}(t_1,\ell(t_1)) > \mathsf{E}(t_1)$ for some t_1. By virtue of the energy balance, $t \mapsto \mathcal{E}(t,\ell(t))$ and $t \mapsto \mathsf{E}(t)$ are continuous and hence the inequality $\mathcal{E}(t,\ell(t)) > \mathsf{E}(t)$ must hold in a non empty interval (t_0,t_1). Taking for t_0 the lowest bound, we have $\mathcal{E}(t_0,\ell(t_0)) = \mathsf{E}(t_0)$. Moreover, $\ell(t) > l(t)$ for all $t \in (t_0,t_1)$. (Indeed, if $\ell(t) \leq l(t)$ at some t, then we should obtain from the second item of the *FM-law* that $\mathcal{E}(t,\ell(t)) \leq \mathcal{E}(t,l(t)) = \mathsf{E}(t)$, what is a contradiction with our first hypothesis.) From the energy balance (satisfied both by $t \mapsto \ell(t)$ and $t \mapsto l(t)$) and the strict monotonicity of P, we get for almost all $t \in (t_0,t_1)$

$$\frac{d}{dt}\mathcal{E}(t,\ell(t)) = 2t\mathsf{P}(\ell(t)) < 2t\mathsf{P}(l(t)) = \frac{d\mathsf{E}}{dt}(t).$$

Integrating over the interval (t_0, t_1), we obtain $\mathcal{E}(t_1, \ell(t_1)) < \mathsf{E}(t_1)$ and hence a contradiction. $\boxdot$

$\square$

Remark 1 *After the change of variable $s = \mathsf{S}(l)$, the minimization problem becomes $\min_{s\in[0,\mathsf{S}(L)]}\{t^2\mathsf{P}\circ\mathsf{S}^{-1}(s)+s\}$. The properties of the solution of this equivalent minimization problem strongly depend on the convexity properties of $\mathsf{P}\circ\mathsf{S}^{-1}$.*

Let us note that the equivalence of the *FM-law* with the minimization problem holds only because we consider an increasing loading. In turn, that ensures the existence of a solution for the *FM-law* and that the crack length will grow from 0 to L without any reference to convexity properties of the energies. It is a first major difference with the *G-law*. The next propositions complete the comparison.

Proposition 4 *The* FM-law *admits a continuous solution if and only if $\ell \mapsto \mathsf{g}(\ell)$ is decreasing. In such a case, the solution is unique and is also the unique solution of the* G-law.

Proof. Let us assume that $t \mapsto \ell(t)$, solution of the *FM-law*, is continuous. Let $0 < \ell_1 < \ell_2 < L$. There exist t_1 and t_2 with $t_1 < t_2$ such that $\ell(t_1) = \ell_1$ and $\ell(t_2) = \ell_2$. By virtue of Proposition 3, ℓ_1 (resp. ℓ_2) minimizes the total energy at t_1 (resp. t_2) over $[0, L]$. Since those points are interior points, the derivative $\partial\mathcal{E}/\partial\ell$ must vanish at (t_1, ℓ_1) (resp. (t_2, ℓ_2)). Hence $1 = t_1^2\mathsf{g}(\ell_1) = t_2^2\mathsf{g}(\ell_2)$. Setting $\mathsf{t_i} = \max\{t : \ell(t) = 0\}$ we obtain by continuity $1 = \mathsf{t_i}^2\mathsf{g}(0)$. Let us set $\mathsf{t_f} = \min\{t : \ell(t) = L\}$ if L is reached at a finite time, and $\mathsf{t_f} = +\infty$ otherwise. In the former case, we obtain by continuity $1 = \mathsf{t_f}^2\mathsf{g}(L)$ from which we deduce that g is decreasing in $[0, L]$. In the latter case, passing to the limit in $\mathsf{g}(\ell(t)) = 1/t^2$ when t goes to ∞, we obtain $\mathsf{g}(L) = 0$ and we still deduce that g is decreasing in $[0, L]$. Moreover, in any case, we have obtained that $\ell(t)$ is the unique solution of the *G-law*, cf. Proposition 2.

Conversely, let us assume that g is decreasing. Then $\mathsf{P}\circ\mathsf{S}^{-1}$ is strictly convex. Therefore, owing to Remark 1, the solution is unique and we easily check that it is continuous and corresponds to the solution of the *G-law*. $\square$

Comparing Propositions 2 and 4 shows that the evolution of the crack length must be discontinuous when $\mathsf{P}\circ\mathsf{S}^{-1}$ is not strictly convex and that the *FM-law* only is able to manage this situation. It is in particular the case when $\mathsf{P}'(0) = 0$. Indeed, then $\mathsf{g}(0) = 0$ and, since $\mathsf{g} \geq 0$, g cannot be decreasing near 0. The following Proposition specifies what happens in such a case.

Proposition 5 *If $\mathsf{P}'(0) = 0$, then, according to whether the line segment joining $(0, \mathsf{P}(0))$ and $(\mathsf{S}(L), \mathsf{P}(L))$ is below the graph of $s \mapsto \mathsf{P}\circ\mathsf{S}^{-1}(s)$, each solution $t \mapsto \ell(t)$ of* FM-law *enjoys the following properties:*

1. *If the line segment is below the graph,*

$$\ell(t) = \begin{cases} 0 & \textit{if} \quad 0 \leq t < \mathsf{t_i} = \sqrt{\frac{\mathsf{S}(L)}{\mathsf{P}(0)-\mathsf{P}(L)}} \\ L & \textit{if} \quad t > \mathsf{t_i} \end{cases}. \tag{10}$$

2. *If the line segment is not below the graph,*
   (a) *There exists $\mathsf{t_i} > 0$ and $\ell_\mathsf{i} \in (0, L)$ such that $\ell(t) = 0$ for $t < \mathsf{t_i}$ and $\ell(\mathsf{t_i}+) = \ell_\mathsf{i}$;*

(b) $\mathsf{t_i}$ *and* ℓ_i *satisfy*

$$\mathsf{P}(0) - \mathsf{P}(\ell_\mathsf{i}) = \mathsf{g}(\ell_\mathsf{i})\mathsf{S}(\ell_\mathsf{i}), \qquad \mathsf{t_i} = \mathsf{g}(\ell_\mathsf{i})^{-1/2} \tag{11}$$

Proof. The proof rests on the fact that $\ell(t)$ is a minimizer of $t^2\mathsf{P}(l) + \mathsf{S}(l)$, P is decreasing and the energies are smooth. It is geometrical by nature, because narrowly related to the energy convexification. All the geometrical objects refer to the plane (s,p), the s-axis corresponding to surface energy and the p-axis to potential energy. The graph of $s \mapsto \mathsf{P} \circ \mathsf{S}^{-1}(s)$ in this plane is referred to as the graph.

Let $\ell \in [0,L]$ and $g \geq 0$. The line (segment) through the point $(\mathsf{S}(\ell), \mathsf{P}(\ell))$ with slope $-g$ is said supporting for the graph at this point if $\mathsf{P}(\ell) - g(\mathsf{S}(l) - \mathsf{S}(\ell)) \leq \mathsf{P}(l)$ for all $l \in [0,L]$. (In other words, the line is below the graph and they have the point as a common point.) Let us first remark that $l \in [0,L]$ is a minimizer of the energy at some $t > 0$ if and only if the line through $(\mathsf{S}(l), \mathsf{P}(l))$ with slope $-1/t^2$ is a supporting line. (It is a simple rephrasing of the optimality condition.) Moreover, if $l \in (0,L)$ is a minimizer of the energy at time t, then the derivative at l of the energy vanishes and hence $t^2\mathsf{g}(l) = 1$. Therefore the line with slope $-\mathsf{g}(l)$ through $(\mathsf{S}(l), \mathsf{P}(l))$ is a supporting line. (The graph is above its tangent.)

Let $\mathcal{M}$ be the set of all the $g \geq 0$ such that the line through $(0, \mathsf{P}(0))$ with slope $-g$ is a supporting line. We have $g \geq (\mathsf{P}(0) - \mathsf{P}(L))/\mathsf{S}(L) > 0$, $\forall g \in \mathcal{M}$ and $\max_{l \in [0,L]} \mathsf{g}(l) \in \mathcal{M}$. Hence $\mathcal{M}$ is a closed nonempty interval of the form $[\mathsf{g}_i, +\infty)$, with

$$\mathsf{g}_i = \min \mathcal{M} \geq \frac{\mathsf{P}(0) - \mathsf{P}(L)}{\mathsf{S}(L)} > 0.$$

Let $\mathsf{t_i} = \mathsf{g}_i^{-1/2}$. When $0 \leq t \leq \mathsf{t_i}$, $l = 0$ is a minimizer and when $0 \leq t < \mathsf{t_i}$ it is the unique minimizer. (Indeed, if another l was a minimizer, then the line through $(0, \mathsf{P}(0))$ with slope $-1/t^2$ should be also a supporting line at $(\mathsf{S}(l), \mathsf{P}(l))$, which is not possible because this point is above the line.) Hence $\ell(t) = 0$ for $t \in [0, \mathsf{t_i})$. At $t = \mathsf{t_i}$, since $\mathsf{P}'(0) = 0$, there exists at least one $l \in (0,L)$ such that the line through $(0, \mathsf{P}(0))$ with slope $-\mathsf{g}_i$ is a supporting line both at $(0, \mathsf{P}(0))$ and at $(\mathsf{S}(l), \mathsf{P}(l))$. The set of all such l is closed, let ℓ_i be its upper bound. All these l's are also minimizers of the energy at $\mathsf{t_i}$.

Let us prove that $\lim_{t \downarrow \mathsf{t_i}} \ell(t) = \ell_\mathsf{i}$. The limit exists because $t \mapsto \ell(t)$ is monotone, say l_i. By continuity, l_i must be a minimizer of the energy at time $\mathsf{t_i}$ and hence $l_i \leq \ell_\mathsf{i}$. If $l_i < \ell_\mathsf{i}$, then there exists an interval $(\mathsf{t_i}, \mathsf{t_i} + h)$ where the line through $(\mathsf{S}(\ell(t)), \mathsf{P}(\ell(t)))$ with slope $-1/t^2$ is above the point $(\mathsf{S}(\ell_\mathsf{i}), \mathsf{P}(\ell_\mathsf{i}))$ what is impossible because it is a supporting line.

Let us now distinguish the two cases.

Case 1. The line segment joining $(0, \mathsf{P}(0))$ and $(\mathsf{S}(L), \mathsf{P}(L))$ is a supporting line at both ends. Therefore $\mathsf{g}_i = (\mathsf{P}(0) - \mathsf{P}(L))/\mathsf{S}(L)$ and $\ell_\mathsf{i} = L$.

Case 2. Then the line segment joining $(0, \mathsf{P}(0))$ and $(\mathsf{S}(L), \mathsf{P}(L))$ is not a supporting line and $0 < \ell_\mathsf{i} < L$. Therefore, $\mathsf{g}_i = \mathsf{g}(\ell_\mathsf{i})$.

Let us note that the proof is a little more precise than the statement, since it gives the exact definition of ℓ_i. □

Remark 2 *In other words, the onset of cracking is necessarily brutal when* $\mathsf{P}'(0) = 0$. *At the critical load when the first cracking occurs, the crack will be initiated on all or a part of the length L. In the latter case, the two equations* (11) *giving the initiation length and the initiation loading can be interpreted as follows:*

1. *The total energy is conserved during the initiation. The release of potential energy exactly supplies the added surface energy;*
2. *The energy release rate* just after *the initiation is equal to the toughness. The Griffith criterion is satisfied in terms of the quantities defined* not before *but* after *the initiation.*

In the second case, what happens after the initiation depends once more on the convexity properties of $\mathsf{P}\circ\mathsf{S}^{-1}$. The following example, which corresponds to frequent practical cases, illustrates what happens when g is first increasing, then decreasing. It is the case in the problem of tearing of a notch considered in the next section, see also [2] or [3] for an application to a fiber debonding.

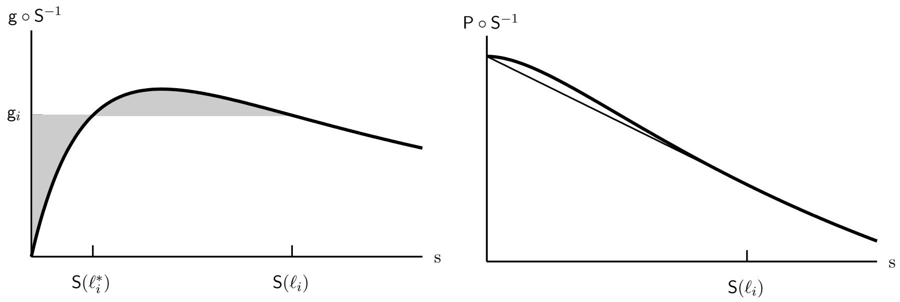

Fig. 1 *Graphical determination of the crack length ℓ_i using the convexification of $\mathsf{P}\circ\mathsf{S}^{-1}$. The slope $\mathsf{g}_i := \mathsf{g}(\ell_\mathsf{i})$ corresponds to the Maxwell line,* i.e. *the line giving the equality of the areas in the graph of $\mathsf{g}\circ\mathsf{S}^{-1}$.*

Proposition 6 *Let g be such that $\mathsf{g}(0) = 0$, g is continuously differentiable in $(0, L)$, $\mathsf{g}' > 0$ in $(0, \ell_c)$, $\mathsf{g}' < 0$ in (ℓ_c, L) and $\mathsf{P}(0) - \mathsf{P}(L) > \mathsf{g}(L)\mathsf{S}(L)$, with $0 < \ell_c < L$. Let ℓ_i be the unique solution of*

$$\ell_c < \ell_\mathsf{i} < L, \qquad \mathsf{P}(0) = \mathsf{P}(\ell_\mathsf{i}) + \mathsf{g}(\ell_\mathsf{i})\mathsf{S}(\ell_\mathsf{i}). \tag{12}$$

Then the solution of the FM-law *enjoys the following properties*

1. *As long as $0 \le t < \mathsf{t_i} := \mathsf{g}(\ell_\mathsf{i})^{-1/2}$, $\ell(t) = 0$;*
2. *At $t = \mathsf{t_i}$, the crack length jumps from 0 to ℓ_i;*
3. *When $\mathsf{t_i} < t \le \mathsf{g}(L)^{-1/2}$, the crack length grows continuously from ℓ_i to L, $t \mapsto \ell(t)$ satisfying then the* G-law*: $t^2\mathsf{g}(\ell(t)) = 1$.*

Proof. We give a direct proof without referring to the previous Proposition. Let us first prove that there exists a unique ℓ_i satisfying (12). Let $\phi(\ell) := \mathsf{P}(\ell) - \mathsf{P}(0) + \mathsf{g}(\ell)\mathsf{S}(\ell)$. Then, since $\phi' = \mathsf{g}'\mathsf{S}$, ϕ is increasing in $(0, \ell_c)$ and decreasing in (ℓ_c, L). Since $\phi(0) = 0$ and since $\phi(L) < 0$ (by hypothesis), there exists a unique $\ell \in (\ell_c, L)$ such that $\phi(\ell) = 0$. The initiation length ℓ_i may be graphically determined, either by drawing the convexification of $\mathsf{P}\circ\mathsf{S}^{-1}$ or by drawing the Maxwell line which satisfies the rule of equality of areas in the graph of $\mathsf{g}\circ\mathsf{S}^{-1}$, see Figure 1.

Thanks to the hypotheses, we get that, for all $l \in [0, L]$, $\mathsf{t_i}^2(\mathsf{P}(l) - \mathsf{P}(0)) + \mathsf{S}(l) \ge 0$ and that the equality holds if and only if $l \in \{0, \ell_\mathsf{i}\}$. Consequently, for $0 \le t < \mathsf{t_i}$ and

$0 \leq l \leq L$, we have $t^2(\mathsf{P}(l) - \mathsf{P}(0)) + \mathsf{S}(l) \geq 0$ and the equality holds if and only if $l = 0$. Hence, $l = 0$ is the unique minimizer when $0 \leq t < \mathsf{t_i}$, while $l = 0$ and $l = \ell_\mathsf{i}$ are the two minimizers when $t = \mathsf{t_i}$. For $t > \mathsf{t_i}$, since $t^2\mathsf{P}(0) > t^2\mathsf{P}(\ell_\mathsf{i}) + \mathsf{S}(\ell_\mathsf{i})$, $l = 0$ is no more a minimizer. For $t < \mathsf{g}(L)^{-1/2}$, since $t^2\mathsf{P}'(L) + \mathsf{S}'(L) > 0$, $l = L$ is not a minimizer. Therefore, when $\mathsf{t_i} < t < \mathsf{g}(L)^{-1/2}$, the minimizer l must be in the open interval $(0, L)$ and hence such that $t^2\mathsf{g}(l) = 1$. Among the two possible solutions, only that greater than ℓ_i is relevant. □

3 Tearing of a notch-shaped body

The preceding general analysis is applied to a concrete case. By sake of simplicity of the presentation and of the proofs, we consider a problem where both the geometry and the loading are simple. It is sufficient to illustrate the fundamental differences between the two laws of propagation. It also points out all the mathematical work necessary to check out the relevant properties of the energy.

3.1 Hypotheses and definitions

3.1.1 Setting of the problem

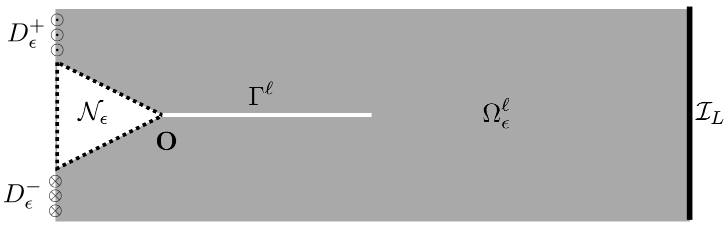

Fig. 2 The notch-shaped body with a crack of length ℓ

Let Ω be the rectangle $(-H, L) \times (-H, +H)$, Ω is the natural configuration of a brittle isotropic body with shear modulus μ and toughness G_c. A symmetric triangular sector $\mathcal{N}_\epsilon$ with an angle $\omega = 2 \arctan \epsilon$, $0 \leq \epsilon < 1$, at $\mathbf{O} = (0, 0)$ is removed from Ω so that the resulting body is the notch-shaped body Ω_ϵ, cf Figure 2:

$$\mathcal{N}_\epsilon = \{\mathbf{x} = (x_1, x_2) : -H < x_1 \leq 0, |x_2| \leq \epsilon\, |x_1|\}, \quad \Omega_\epsilon = \Omega \setminus \mathcal{N}_\epsilon. \tag{13}$$

The shape ratio ϵ of the notch will be considered as a parameter of the problem. The case $\epsilon = 0$ corresponds to a body with an initial crack of length H

$$\mathcal{N}_0 = (-H, 0] \times \{0\}, \quad \Omega_0 = \Omega \setminus \mathcal{N}_0. \tag{14}$$

The body is submitted to an anti-plane loading so that the displacements of the body are orthogonal to the plane (x_1, x_2). Specifically, the anti-plane displacement component u is prescribed on the parts $D_\epsilon^\pm$ of the boundary:

$$u = \pm t \quad \text{on } D_\epsilon^\pm, \qquad D_\epsilon^+ = \{-H\} \times (\epsilon H, H), \quad D_\epsilon^- = \{-H\} \times (-H, -\epsilon H), \tag{15}$$

whereas the end $\mathcal{I}_L = \{L\} \times (-H, H)$ is fixed, *i.e.* $u = 0$ on $\mathcal{I}_L$. The remaining parts of the boundary (and in particular the edges of the notch) are free. The body forces are neglected. Owing to the symmetry of the geometry and the loading, we assume that a crack will appear at the tip of the notch and will propagate inside the body along the predefined straight path $\Gamma = (0, L] \times \{0\}$. When the length of the crack is ℓ, $0 < \ell \leq L$, we denote by Γ^ℓ the (add-)crack and by Ω_ϵ^ℓ the resulting cracked notch-shaped body

$$\Gamma^\ell = (0, \ell) \times \{0\}, \qquad \Omega_\epsilon^\ell = \Omega_\epsilon \setminus \Gamma^\ell. \tag{16}$$

At a loading t and a crack length ℓ corresponds the displacement field $u_\epsilon^\ell(\mathbf{x}, t)$ of the cracked body Ω_ϵ^ℓ at equilibrium under that loading. By linearity, $u_\epsilon^\ell(\mathbf{x}, t)$ is proportional to t and can read as $u_\epsilon^\ell(\mathbf{x}, t) = tU_\epsilon^\ell(\mathbf{x})$ where the field U_ϵ^ℓ has to satisfy

$$\begin{cases} \Delta U_\epsilon^\ell = 0 & \text{in} \quad \Omega_\epsilon^\ell, \\ U_\epsilon^\ell = \pm 1 & \text{on} \quad D_\epsilon^\pm, \\ U_\epsilon^\ell = 0 & \text{on} \quad \mathcal{I}_L, \\ \frac{\partial U_\epsilon^\ell}{\partial n} = 0 & \text{on} \quad (\partial\Omega_\epsilon \cup \Gamma^\ell) \setminus (D_\epsilon^+ \cup D_\epsilon^- \cup \mathcal{I}_L) \end{cases} . \tag{17}$$

The first equation of (17) corresponds to the bulk equilibrium equation, the last one contains the condition of non cohesive crack lips.

Remark 3 *The displacement field U_ϵ^ℓ can be seen as the real part of a complex potential f_ϵ^ℓ holomorphic in Ω_ϵ^ℓ, see [34]. The imaginary part ψ_ϵ^ℓ of f_ϵ^ℓ corresponds to the Airy function related to the stress field $\boldsymbol{\sigma}_\epsilon^\ell := \mu\nabla U_\epsilon^\ell$ by $\sigma_{\epsilon 1}^\ell = \mu\psi_{\epsilon,2}^\ell$, $\sigma_{\epsilon 2}^\ell = -\mu\psi_{\epsilon,1}^\ell$, ψ_ϵ^ℓ is also harmonic in Ω_ϵ^ℓ. The Neumann boundary conditions $\nabla U_\epsilon^\ell \cdot \mathbf{n} = 0$ read as $\nabla\psi_\epsilon^\ell \cdot \mathbf{t} = 0$ in terms of the Airy function (where* $\mathbf{n}$ *and* $\mathbf{t}$ *denotes the outer unit normal and the unit tangent vector to the boundary). Hence, by fixing the arbitrary constant of ψ_ϵ^ℓ, the boundary conditions on the lips of the crack and on the edges of the notch can be written $\psi_\epsilon^\ell = 0$.*

Because of the symmetry of the geometry and of the loading, U_ϵ^ℓ is a odd function of x_2, $U_\epsilon^\ell(x_1, -x_2) = -U_\epsilon^\ell(x_1, x_2)$, and hence $U_\epsilon^\ell(x_1, 0) = 0$ when $\ell \leq x_1 \leq L$. However, we will essentially use this property for the numerical computations only.

The cracked or uncracked notch-shaped body contains several corners where the displacement is a priori singular. Let us give the comprehensive list of such points with their associated singularity by using Grisvard's results [23, 24].

1. *At the tip of the notch* $\mathbf{O}$*, when there is no (add)-crack, the displacement is a priori singular and can read as*

$$U_\epsilon^0(\mathbf{x}) = K_\epsilon r^{\alpha_\epsilon} \sin(\alpha_\epsilon\theta) + \underline{U_\epsilon^0}(\mathbf{x}) \tag{18}$$

where (r, θ) denote the polar coordinates, $\alpha_\epsilon = \pi/(2\pi - 2\arctan\epsilon)$ and $\underline{U_\epsilon^0} \in H^2(\Omega_\epsilon \cap B_r)$ for r small enough.

2. *At the tip* $(\ell, 0)$ *of the crack, when* $0 < \ell < L$*, the displacement is a priori singular and can read as*
$$U_\epsilon^\ell(\mathbf{x}) = K_\epsilon^\ell \sqrt{r} \sin\frac{\theta}{2} + \underline{U_\epsilon^\ell}(\mathbf{x}) \tag{19}$$
where (r, θ) *denote the polar coordinates with pole* $(\ell, 0)$ *and* $\underline{U_\epsilon^\ell} \in H^2(\Omega_\epsilon^\ell \cap B_r)$ *for* r *small enough. The coefficient* K_ϵ^ℓ *is the stress intensity factor.*
3. *When* $\epsilon > 0$*, at the corners of the notch* $(-H, \pm\epsilon H)$*, because of the change of boundary conditions from Dirichlet to Neumann at an angle greater than* $\pi/2$*, the displacement is a priori singular, but this singularity will play no role.*
4. *At the corners* $(-H, \pm H)$ *and* $(L, \pm H)$*, at* $(L, 0)$ *when* $\ell = L$*, and at* $(-H, 0\pm)$ *when* $\epsilon = 0$*, there is no singularity (that means that* U_ϵ^ℓ *is locally in* H^2*) even if there is a change of boundary conditions. But, since the angle is equal to* $\pi/2$*, we are just at the limit to have a singularity.*
5. *At the corners of the notch* $(0, 0\pm)$*, once a crack has been created, there is no more singularity because the angle is less than* π *with Neumann boundary conditions.*

The elastic energy of the cracked body is given by
$$\mathcal{P}_\epsilon(t, \ell) = t^2 \mathsf{P}_\epsilon(\ell), \qquad \mathsf{P}_\epsilon(\ell) := \int_{\Omega_\epsilon^\ell} \frac{\mu}{2} \nabla U_\epsilon^\ell \cdot \nabla U_\epsilon^\ell \, dx$$
and, by virtue of the theorem of potential energy minimum, we have
$$\mathsf{P}_\epsilon(\ell) = \min_{u \in \mathcal{U}_\epsilon^\ell} \int_{\Omega_\epsilon^\ell} \frac{\mu}{2} \nabla u \cdot \nabla u \, dx. \tag{20}$$
In (20), $\mathcal{U}_\epsilon^\ell$ is the set of admissible displacements,
$$\mathcal{U}_\epsilon^\ell = \{u \in H^1(\Omega_\epsilon^\ell) : u = \pm 1 \text{ on } D_\epsilon^\pm, \quad u = 0 \text{ on } \mathcal{I}_L\}.$$
It is a closed affine subset of $H^1(\Omega_\epsilon^\ell)$, its associated linear subspace is
$$\mathcal{V}_\epsilon^\ell = \{v \in H^1(\Omega_\epsilon^\ell) : v = 0 \text{ on } D_\epsilon^+ \cup D_\epsilon^- \cup \mathcal{I}_L\}.$$
Let $\overline{U}_0$ be the field defined on Ω_0 by $\overline{U}_0(\mathbf{x}) = \operatorname{sign}(x_2)\max\{-x_1/H, 0\}$ and $\overline{U}_\epsilon$ its restriction to Ω_ϵ, $\overline{U}_\epsilon \in \mathcal{U}_\epsilon^\ell$ for every $\epsilon \in [0, 1)$. Decomposing $u \in \mathcal{U}_\epsilon^\ell$ into $u = \overline{U}_\epsilon + v$ with $v \in \mathcal{V}_\epsilon^\ell$, the potential energy can also read as
$$\mathsf{P}_\epsilon(\ell) = \min_{v \in \mathcal{V}_\epsilon^\ell} \left\{ \int_{\Omega_\epsilon^\ell} \frac{\mu}{2} \nabla v \cdot \nabla v \, dx - \int_{\mathcal{N}_\epsilon^c} \operatorname{sign}(x_2) \frac{\mu}{H} v_{,1} \, dx + \mu(1 - \frac{\epsilon}{2}) \right\} \tag{21}$$
where $\mathcal{N}_\epsilon^c = (-H, 0] \times (-H, H) \setminus \mathcal{N}_\epsilon$. The minimizer in (21) is $V_\epsilon^\ell = U_\epsilon^\ell - \overline{U}_\epsilon$.

3.1.2 Notations

Note that t has the dimension of a length, hence U_ϵ^ℓ is dimensionless and P_ϵ has the dimension of a pressure. Throughout the section we will use the following notations. We denote by $\mathcal{I}_l$ the cross-section of the (sound) body at $l \in [0, L]$, *i.e.* $\mathcal{I}_l = \{l\} \times (-H, H)$ and by $\mathcal{R}_l^d$ the rectangle delimited by the cross-sections $\mathcal{I}_l$ and $\mathcal{I}_d$, $0 \le l < d \le L$, *i.e.* $\mathcal{R}_l^d = (l, d) \times (-H, H)$. The "cracked" cross-section at l is denoted $\underline{\mathcal{I}_l}$, *i.e.* $\underline{\mathcal{I}_l} = \{l\} \times ((-H, 0) \cup (0, H))$, and the cracked rectangle between $\mathcal{I}_l$ and $\mathcal{I}_d$ is denoted $\underline{\mathcal{R}_l^d}$, *i.e.* $\underline{\mathcal{R}_l^d} = (l, d) \times ((-H, 0) \cup (0, H))$. If $\mathcal{D}$ is a sub-domain of Ω and v is a real-valued field on $\mathcal{D}$, $\|v\|_\mathcal{D}$ stands for the $H^1(\mathcal{D})$ norm of v whereas $|v|_\mathcal{D}$ stands for its $L^2(\mathcal{D})$ norm. B_r denotes the ball of center $\mathbf{O}$ and radius $r > 0$.

3.2 Check of the regularity of $\mathsf{P}_\epsilon(\ell)$ with respect to ℓ

3.2.1 Case $0 \leq \epsilon < 1$ *and* $0 < \ell < L$

Throughout this subsection, ϵ is fixed and the explicit dependence on it is sometimes omitted. First, we make a change of variables to send the ℓ-dependent domain Ω_ϵ^ℓ onto a fix domain. In essence, it is the basic method to prove the existence of the energy release rate, see [14]. In our case, owing to the simplicity of the geometry, we can use a very simple change of variable which renders the proof easier. Furthermore, that allows for obtaining stronger regularity results.

Specifically, let us choose $d \in (0, L)$ and let ϕ_ℓ be the following lipschitzian homeomorphism from Ω_ϵ^ℓ onto Ω_ϵ^d

$$\widetilde{\mathbf{x}} := \phi_\ell(\mathbf{x}) = \mathbf{x} + \begin{cases} \mathbf{0} & \text{if } x_1 < 0 \\ (d-\ell)\frac{x_1}{\ell}\mathbf{e}_1 & \text{if } 0 \leq x_1 \leq \ell \\ (d-\ell)\frac{L-x_1}{L-\ell}\mathbf{e}_1 & \text{if } \ell \leq x_1 < L \end{cases} . \tag{22}$$

With each field v defined on Ω_ϵ^ℓ is associated its push-forward $\widetilde{v} = v \circ \phi_\ell^{-1}$ defined on Ω_ϵ^d, see [32]. Hence, $\widetilde{v} \in \mathcal{V}_\epsilon^d$ if and only if $v \in \mathcal{V}_\epsilon^\ell$. Noting that

$$\nabla v \cdot \nabla v \ \det F_\ell \ dx = F_\ell^T \nabla \widetilde{v} \cdot F_\ell^T \nabla \widetilde{v} \ d\widetilde{x}$$

with

$$F_\ell := \nabla \phi_\ell = \mathbf{e}_2 \otimes \mathbf{e}_2 + \begin{cases} \mathbf{e}_1 \otimes \mathbf{e}_1 & \text{if } x_1 < 0 \\ \frac{d}{\ell}\mathbf{e}_1 \otimes \mathbf{e}_1 & \text{if } 0 \leq x_1 \leq \ell \\ \frac{L-d}{L-\ell}\mathbf{e}_1 \otimes \mathbf{e}_1 & \text{if } \ell \leq x_1 < L \end{cases}$$

and inserting this change of variable into (21) leads to

$$\mathsf{P}_\epsilon(\ell) = \min_{\widetilde{v} \in \mathcal{V}_\epsilon^d} \left\{ \frac{1}{2} \sum_{i=1}^{5} a_\ell^i \mathsf{p}^i(\widetilde{v}, \widetilde{v}) + \mathsf{q}(\widetilde{v}) + c \right\} \tag{23}$$

with

$$\begin{cases} a_\ell^1 = 1 & , \quad \mathsf{p}^1(u,v) = \int_{\mathcal{N}_\epsilon^c} \mu \nabla u \cdot \nabla v \ dx \\ a_\ell^2 = \frac{d}{\ell} & , \quad \mathsf{p}^2(u,v) = \int_{\mathcal{R}_0^d} \mu u_{,1} v_{,1} \ dx \\ a_\ell^3 = \frac{\ell}{d} & , \quad \mathsf{p}^3(u,v) = \int_{\mathcal{R}_0^d} \mu u_{,2} v_{,2} \ dx \\ a_\ell^4 = \frac{L-d}{L-\ell} & , \quad \mathsf{p}^4(u,v) = \int_{\mathcal{R}_d^L} \mu u_{,1} v_{,1} \ dx \\ a_\ell^5 = \frac{L-\ell}{L-d} & , \quad \mathsf{p}^5(u,v) = \int_{\mathcal{R}_d^L} \mu u_{,2} v_{,2} \ dx \\ c = (1 - \frac{\epsilon}{2})\mu \ , & \quad \mathsf{q}(v) = -\int_{\mathcal{N}_\epsilon^c} \operatorname{sign}(x_2) \frac{\mu}{H} v_{,1} \ dx \end{cases} \tag{24}$$

Note that $\mathsf{q}(v) = \mathsf{p}^1(\overline{U}_\epsilon, v)$ and $2c = \mathsf{p}^1(\overline{U}_\epsilon, \overline{U}_\epsilon)$. The minimizer in (23) is $\widetilde{V_\epsilon^\ell}$, the push-forward of V_ϵ^ℓ, and $\widetilde{V_\epsilon^\ell} = \widetilde{U_\epsilon^\ell} - \overline{U}_\epsilon$ because $\widetilde{\overline{U}_\epsilon} = \overline{U}_\epsilon$. We are now in position to prove the first regularity result

Proposition 7 *For each* $\epsilon \in [0, 1)$, $\ell \mapsto \mathsf{P}_\epsilon(\ell)$ *is indefinitely differentiable on* $(0, L)$. *Moreover the first derivative* P'_ϵ *can read as*

$$\mathsf{P}'_\epsilon(\ell) = \frac{1}{\ell} \int_{\mathcal{R}_0^\ell} \frac{\mu}{2} \Big(\big(U_{\epsilon,2}^\ell\big)^2 - \big(U_{\epsilon,1}^\ell\big)^2 \Big) dx - \frac{1}{L-\ell} \int_{\mathcal{R}_\ell^L} \frac{\mu}{2} \Big(\big(U_{\epsilon,2}^\ell\big)^2 - \big(U_{\epsilon,1}^\ell\big)^2 \Big) dx \tag{25}$$

Proof. Since the five a^i_ℓ are indefinitely differentiable on $(0, L)$, we can use Lemma 18 and Remark 6 of the Appendix with $\lambda = \ell$, $\Lambda = (0, L)$ and $\mathcal{H} = \mathcal{V}^d_\epsilon$ equipped with the norm of $H^1(\Omega^d_\epsilon)$. (The uniform coercivity of $\sum_{i=1}^5 a^i_\ell \mathsf{p}^i$ holds in any closed interval included in $(0, L)$ by virtue of Poincaré's inequality; the continuity of the p^i_ℓ's and q is obvious.) That proves the regularity of P_ϵ.

Using (60) gives $\mathsf{P}'_\epsilon(\ell) = \frac{1}{2}\sum_{i=2}^5 \dot{a}^i_\ell \mathsf{p}^i(\widetilde{U^\ell_\epsilon}, \widetilde{U^\ell_\epsilon})$, because $\dot{a}^1_\ell = 0$ and $\overline{U}_\epsilon$ vanishes when $x_1 \geq 0$. Therefore,

$$\mathsf{P}'_\epsilon(\ell) = \frac{1}{d}\int_{\underline{\mathcal{R}}^d_0} \frac{\mu}{2}\Big(\big(\widetilde{U^\ell_\epsilon}_{,2}\big)^2 - \frac{d^2}{\ell^2}\big(\widetilde{U^\ell_\epsilon}_{,1}\big)^2\Big)dx - \frac{1}{L-d}\int_{\mathcal{R}^L_d} \frac{\mu}{2}\Big(\big(\widetilde{U^\ell_\epsilon}_{,2}\big)^2 - \frac{(L-d)^2}{(L-\ell)^2}\big(\widetilde{U^\ell_\epsilon}_{,1}\big)^2\Big)dx \tag{26}$$

Making the inverse change of variable $\mathbf{x} \mapsto \phi^{-1}_\ell(\mathbf{x})$ in the integrals leads to (25). □

The formula (25) is a particular case of the one proposed by [14] to compute the energy release rate. This integral over the domain is related to the famous path-independent integrals (see [39]) by virtue of the following

Proposition 8 *Let $\mathcal{I}$ be a cross-section of Ω^ℓ_ϵ and $\mathcal{J}_\mathcal{I} := \int_\mathcal{I} \frac{\mu}{2}\Big(\big(U^\ell_{\epsilon,2}\big)^2 - \big(U^\ell_{\epsilon,1}\big)^2\Big)dx_2$. Then $\mathcal{J}_{\underline{\mathcal{I}}_l}$ is independent of l for $l \in (0, \ell)$, the common value is denoted $\mathsf{J}^-_\epsilon(\ell)$. Similarly, $\mathcal{J}_{\mathcal{I}_l}$ is independent of l for $l \in (\ell, L)$, the common value is denoted $\mathsf{J}^+_\epsilon(\ell)$. Therefore $\mathsf{P}'_\epsilon(\ell) = \mathsf{J}^-_\epsilon(\ell) - \mathsf{J}^+_\epsilon(\ell)$ for $\ell \in (0, L)$.*

Proof. Note first that $\mathcal{J}_{\underline{\mathcal{I}}_l}$ (resp. $\mathcal{J}_{\mathcal{I}_l}$) is well-defined for every $l \in (0, \ell)$ (resp. $l \in (\ell, L)$) because $U^\ell_\epsilon \in H^2(\underline{\mathcal{R}}^{l+h}_{l-h})$ (resp. $U^\ell_\epsilon \in H^2(\mathcal{R}^{l+h}_{l-h})$) for h small enough. Let $0 < l_1 < l_2 < \ell$, multiply the equilibrium equation $0 = \mu\Delta U^\ell_\epsilon$ by $U^\ell_{\epsilon,1}$ and integrate over $\underline{\mathcal{R}}^{l_2}_{l_1}$. Integrating by parts and accounting for the boundary conditions on the lips of the crack and on the upper and lower sides of the rectangle lead to

$$0 = -\int_{\underline{\mathcal{R}}^{l_2}_{l_1}} \mu U^\ell_{\epsilon,i} U^\ell_{\epsilon,i1}\, dx + \int_{\underline{\mathcal{I}}_{l_2}} \mu\big(U^\ell_{\epsilon,1}\big)^2 dx_2 - \int_{\underline{\mathcal{I}}_{l_1}} \mu\big(U^\ell_{\epsilon,1}\big)^2 dx_2.$$

Since the first integrand can read as $\frac{\mu}{2}\big(U^\ell_{\epsilon,i}U^\ell_{\epsilon,i}\big)_{,1}$, we get $0 = \mathcal{J}_{\underline{\mathcal{I}}_{l_1}} - \mathcal{J}_{\underline{\mathcal{I}}_{l_2}}$.

One proves in the same manner that $\mathcal{J}_{\mathcal{I}_{l_2}} = \mathcal{J}_{\mathcal{I}_{l_1}}$ when $\ell < l_1 < l_2 < L$. Inserting into (25) gives $\mathsf{P}'_\epsilon(\ell) = \mathsf{J}^-_\epsilon(\ell) - \mathsf{J}^+_\epsilon(\ell)$. □

The energy release rate $\mathsf{G}_\epsilon(\ell) = -\mathsf{P}'_\epsilon(\ell)$ is related to the stress intensity factor K^ℓ_ϵ introduced in (19) by the well-known Irwin formula, see [25], [28]. In the present setting where there is no normalization of K^ℓ_ϵ, that leads to the relation

$$\mathsf{G}_\epsilon(\ell) = \frac{\pi}{4}\mu(K^\ell_\epsilon)^2. \tag{27}$$

3.2.2 Case $\epsilon = 0$ and $\ell = 0$

The change of variable (22) is no more valid when $\ell = 0$. However, when $\epsilon = 0$, *i.e.* when the body contains an initial crack of length H instead of a notch, we can define P_0 on the whole interval $(-H, L)$ and prove its regularity. Indeed, it is enough to replace (22) by

$$\varphi_\ell(\mathbf{x}) = \mathbf{x} - \begin{cases} \ell\frac{H+x_1}{H+\ell}\mathbf{e}_1 & \text{if } -H \leq x_1 \leq \ell \\ \ell\frac{L-x_1}{L-\ell}\mathbf{e}_1 & \text{if } \ell \leq x_1 < L \end{cases}. \tag{28}$$

Therefore, φ_ℓ is a lipschitzian homeomorphism from Ω_0^ℓ onto Ω_0 for every $\ell \in (-H, L)$. We can use Lemma 18 and Remark 6 again, and with only minor changes in the proof which are not detailed here, we finally obtain

Proposition 9 P_0 *is indefinitely differentiable on* $(-H, L)$. *Moreover the first derivative* P_0' *can read as*

$$\mathsf{P}_0'(\ell) = \frac{1}{\ell+H}\int_{\mathcal{R}_{-H}^{\ell}} \frac{\mu}{2}\Big(\big(U_{0,2}^\ell\big)^2 - \big(U_{0,1}^\ell\big)^2\Big)dx - \frac{1}{L-\ell}\int_{\mathcal{R}_\ell^L} \frac{\mu}{2}\Big(\big(U_{0,2}^\ell\big)^2 - \big(U_{0,1}^\ell\big)^2\Big)dx \tag{29}$$

Remark 4 *As in Proposition 8,* (29) *can be simplified by using the path-independent integrals* $\mathsf{J}_\epsilon^+(\ell)$ *and* $\mathsf{J}_\epsilon^-(\ell)$. *We can also use Irwin's formula* (27) *for all* $\ell \in (-H, L)$. *Note in particular that the positivity of* $\mathsf{G}_0(0)$ *is equivalent to the non vanishing of the stress intensity factor* K_0^0.

3.2.3 Case $0 < \epsilon < 1$ *and* $\ell = 0$

When $\epsilon \neq 0$ and $\ell = 0$, the change of variable (22) is not valid and we cannot extend it like in the previous subsection. We will directly prove that $\mathsf{P}_\epsilon'(0)$ exists and vanishes with the help of classical variational properties. But the proof of the continuity of the derivative at 0 needs a change of variable again. We have the following fundamental result

Proposition 10 *The release of potential energy due to a crack of small length* ℓ *is of the order of* $\ell^{2\alpha_\epsilon}$, i.e. *there exists* $C_\epsilon \geq 0$ *such that*

$$0 \leq \limsup_{\ell \downarrow 0} \frac{\mathsf{P}_\epsilon(0) - \mathsf{P}_\epsilon(\ell)}{\ell^{2\alpha_\epsilon}} \leq C_\epsilon. \tag{30}$$

Therefore $\mathsf{P}_\epsilon'(0) = 0$. *Moreover,* P_ϵ' *is continuous at* 0, $\lim_{\ell\downarrow 0} \mathsf{P}_\epsilon'(\ell) = 0$.

Proof. The proof is divided into 4 steps. The first step of the proof can be seen as a particular case of a more general result proved in [7], see also [29] for a formal proof using matched asymptotic expansions. The third step could be deduced from [11].
Step 1 : $\mathsf{P}_\epsilon'(0) = 0$.
Let $\ell > 0$ be small enough. By virtue of classical duality properties, see [16], we have

$$\begin{aligned}
\mathsf{P}_\epsilon(\ell) &= \min_{u\in\mathcal{U}_\epsilon^\ell} \int_{\Omega_\epsilon^\ell} \frac{\mu}{2}\nabla u.\nabla u\, dx \\
&= \min_{u\in\mathcal{U}_\epsilon^\ell} \max_{\boldsymbol{\tau}\in L^2(\Omega_\epsilon^\ell;\mathbb{R}^2)} \int_{\Omega_\epsilon^\ell} \left(\boldsymbol{\tau}\cdot\nabla u - \frac{\boldsymbol{\tau}\cdot\boldsymbol{\tau}}{2\mu}\right) dx \\
&= \max_{\boldsymbol{\tau}\in L^2(\Omega_\epsilon^\ell;\mathbb{R}^2)} \inf_{u\in\mathcal{U}_\epsilon^\ell} \int_{\Omega_\epsilon^\ell} \left(\boldsymbol{\tau}\cdot\nabla u - \frac{\boldsymbol{\tau}\cdot\boldsymbol{\tau}}{2\mu}\right) dx \\
&= \max_{\boldsymbol{\tau}\in\mathcal{S}_\epsilon^\ell} \int_{\Omega_\epsilon^\ell} \left(\boldsymbol{\tau}\cdot\nabla U_\epsilon^0 - \frac{\boldsymbol{\tau}\cdot\boldsymbol{\tau}}{2\mu}\right) dx
\end{aligned}$$

where $\mathcal{S}_\epsilon^\ell$ stands for the set of statically admissible stress fields, *i.e.* $\mathcal{S}_\epsilon^\ell = \{\boldsymbol{\tau} \in L^2(\Omega_\epsilon^\ell;\mathbb{R}^2) : \int_{\Omega_\epsilon^\ell} \boldsymbol{\tau}\cdot\nabla v\ dx = 0, \forall v \in \mathcal{V}_\epsilon^\ell\}$ and we use the fact that $U_\epsilon^0 \in \mathcal{U}_\epsilon^\ell$. Let $\boldsymbol{\sigma}_\epsilon = \mu\nabla U_\epsilon^0$. Since $\mathrm{div}\boldsymbol{\sigma}_\epsilon = 0$ in $B_{2\ell}\cap\Omega_\epsilon$ and $\boldsymbol{\sigma}_\epsilon\cdot\mathbf{n} = \sigma_{\epsilon\theta} = 0$ on $\partial\Omega_\epsilon \cap B_{2\ell}$, there exists an Airy function $\psi_\epsilon \in H^1(\Omega_\epsilon \cap B_{2\ell})$ such that $\psi_\epsilon = 0$ on $\partial\Omega_\epsilon \cap B_{2\ell}$ and $r\sigma_{\epsilon r} = \mu\psi_{\epsilon,\theta}$, $\sigma_{\epsilon\theta} = -\mu\psi_{\epsilon,r}$ in $B_{2\ell} \cap \Omega_\epsilon$, see Remark 3. Let us construct a statically admissible stress field $\boldsymbol{\tau}$ as follows:

$$\begin{cases} \boldsymbol{\tau} = \mathbf{0} & \text{in} \quad \Omega_\epsilon \cap B_\ell \\ r\tau_r = f(r)\mu\psi_{\epsilon,\theta},\ \tau_\theta = -(f(r)\mu\psi_\epsilon)_{,r} & \text{in} \quad \Omega_\epsilon \cap (B_{2\ell} \setminus B_\ell) \\ \boldsymbol{\tau} = \boldsymbol{\sigma}_\epsilon & \text{in} \quad \Omega_\epsilon \setminus B_{2\ell} \end{cases},$$

where $f(r) = r/\ell - 1$ for $r \in [\ell, 2\ell]$. One easily verifies that $\tau \in \mathcal{S}_\epsilon^\ell$ (in particular, div $\boldsymbol{\tau} = 0$ in $B_{2\ell} \setminus B_\ell$, τ_r is continuous at $r = \ell$ and $r = 2\ell$, $\tau_\theta = 0$ on the boundary of the notch). Therefore,

$$\begin{aligned} \mathsf{P}_\epsilon(0) - \mathsf{P}_\epsilon(\ell) &\le \int_{\Omega_\epsilon^\ell \cap B_{2\ell}} \frac{\|\boldsymbol{\sigma}_\epsilon - \boldsymbol{\tau}\|^2}{2\mu}\, dx \\ &\le \int_{\Omega_\epsilon^\ell \cap B_\ell} \frac{\|\boldsymbol{\sigma}_\epsilon\|^2}{2\mu}\, dx + \int_{\Omega_\epsilon^\ell \cap (B_{2\ell} \setminus B_\ell)} \frac{\sigma_{\epsilon r}^2 + \left(|\sigma_{\epsilon\theta}| + \frac{\mu\psi_\epsilon}{\ell}\right)^2}{2\mu}\, dx. \end{aligned} \quad (31)$$

By virtue of Remark 3 and (18), $\boldsymbol{\sigma}_\epsilon$ and ψ_ϵ behave like $r^{\alpha_\epsilon - 1}$ and r_ϵ^α, respectively, near $r = 0$. (Specifically, the singular part of ψ_ϵ is $K_\epsilon r^{\alpha_\epsilon} \cos\alpha_\epsilon\theta$.) Hence, both integrals in (31) are of the order of $\ell^{2\alpha_\epsilon}$ and (30) follows. Since $\alpha_\epsilon > 1/2$, we get $\mathsf{P}_\epsilon'(0) = 0$. ⊡

<u>Step 2</u> : *Transport of U_ϵ^0 into $\mathcal{U}_\epsilon^d$* .

For $\ell > 0$, the push-forward $\widetilde{U_\epsilon^\ell}$ of U_ϵ^ℓ is an element of $\mathcal{U}_\epsilon^d = \overline{U}_\epsilon + \mathcal{V}_\epsilon^d$. To compare it with U_ϵ^0, we must transport this latter one into $\mathcal{U}_\epsilon^d$. We proceed as follows. Let $\phi_0 : \mathcal{R}_0^L \to \mathcal{R}_d^L$ be the linear one-to-one mapping such that $\phi_0(\mathbf{x}) = (d + (1 - d/L)x_1)\mathbf{e}_1 + x_2\mathbf{e}_2$. With $v : \Omega_\epsilon \to \mathbb{R}$ is associated $\widehat{v} : \Omega_\epsilon^d \to \mathbb{R}$ by

$$\widehat{v}(\mathbf{x}) = \begin{cases} v(\mathbf{x}) & \text{in} \quad \mathcal{N}_\epsilon^c \\ v(0, x_2) & \text{in} \quad \underline{\mathcal{R}}_0^d \\ v \circ {\phi_0}^{-1}(\mathbf{x}) & \text{in} \quad \mathcal{R}_d^L \end{cases}. \quad (32)$$

The image of $\mathcal{V}_\epsilon^0$ by this isomorphism is $\widehat{\mathcal{V}_\epsilon^0}$. Note that, in general, $\widehat{v} \notin \mathcal{V}_\epsilon^d$ even if $v \in \mathcal{V}_\epsilon^0$ because $\widehat{v}_{,2} \notin L^2(\underline{\mathcal{R}}_0^d)$. The elements of $\widehat{\mathcal{V}_\epsilon^0}$ which are in $\mathcal{V}_\epsilon^d$ constitute its (weakly) closed subspace $\hat{\mathcal{V}_\epsilon^0} = \{v \in \mathcal{V}_\epsilon^d : v_{,1} = 0 \text{ in } \underline{\mathcal{R}}_0^d\}$. However, $\widehat{V_\epsilon^0} \in \hat{\mathcal{V}_\epsilon^0}$ because the singularity of V_ϵ^0 at $\mathbf{O}$ is weak. (Indeed, we have to check that $\int_{\underline{\mathcal{I}}_0} \left({V_\epsilon^0}_{,2}\right)^2 dx_2 < +\infty$ so that $\widehat{V_\epsilon^0} \in \mathcal{V}_\epsilon^d$. That is true because ${V_\epsilon^0}_{,2}$ behaves like $|x_2|^{2\alpha_\epsilon - 2}$ with $\alpha_\epsilon > 1/2$.) We still have $\widehat{U_\epsilon^0} = \overline{U}_\epsilon + \widehat{V_\epsilon^0}$. ⊡

<u>Step 3</u> : *Convergence of $\widetilde{U_\epsilon^\ell}$ to $\widehat{U_\epsilon^0}$ when $\ell \to 0$.*

Inserting the change of variable (32) into (21) leads to

$$\mathsf{P}_\epsilon(0) = \min_{\widehat{v} \in \widehat{\mathcal{V}_\epsilon^0}} \left\{ \frac{1}{2}\mathsf{p}^1(\widehat{v} + \overline{U}_\epsilon, \widehat{v} + \overline{U}_\epsilon) + \frac{1}{2}\sum_{i=4}^{5} a_0^i \mathsf{p}^i(\widehat{v}, \widehat{v}) \right\} \quad (33)$$

where the p^i's are the same as in (24), whereas the a_0^i's are obtained by setting $\ell = 0$. The minimizer in (33) is $\widehat{V_\epsilon^0}$ and $\widehat{U_\epsilon^0}$ satisfies

$$\forall \widehat{v} \in \widehat{\mathcal{V}_\epsilon^0}, \quad 0 = \mathsf{p}^1(\widehat{U_\epsilon^0}, \widehat{v}) + a_0^4 \mathsf{p}^4(\widehat{U_\epsilon^0}, \widehat{v}) + a_0^5 \mathsf{p}^1(\widehat{U_\epsilon^0}, \widehat{v}). \quad (34)$$

We deduce from (34) that $\widehat{U^0_\epsilon}$ satisfies the "push-forward" equilibrium equations

$$\Delta\widehat{U^0_\epsilon} = 0 \text{ in } \mathcal{N}^c_\epsilon, \quad a^4_0\widehat{U^0_\epsilon}_{,11} + a^5_0\widehat{U^0_\epsilon}_{,22} = 0 \text{ in } \mathcal{R}^L_d$$

while the continuity of the normal stress $\sigma^0_1 := \mu\, U^0_{\epsilon,1}\big|_{\mathcal{I}_0}$ on $\mathcal{I}_0$ reads now

$$\sigma^0_1(x_2) = \mu\widehat{U^0_\epsilon}_{,1}(0-, x_2) = \mu a^4_0\widehat{U^0_\epsilon}_{,1}(d+, x_2).$$

Multiplying the equilibrium equations by $v \in \mathcal{V}^d_\epsilon$, integrating over $\mathcal{N}^c_\epsilon \cup \mathcal{R}^L_d$, integrating by parts and accounting for the boundary conditions lead to

$$\mathsf{p}^1(\widehat{U^0_\epsilon}, v) + a^4_0\mathsf{p}^4(\widehat{U^0_\epsilon}, v) + a^5_0\mathsf{p}^5(\widehat{U^0_\epsilon}, v) = \mathsf{q}^0_1(v), \quad \forall v \in \mathcal{V}^d_\epsilon \tag{35}$$

with

$$\mathsf{q}^0_1(v) := -\int_{\underline{\mathcal{R}}^d_0} \sigma^0_1 v_{,1} dx. \tag{36}$$

Since $\sigma^0_1 \in L^2(\underline{\mathcal{I}}_0)$, $|\mathsf{q}^0_1(v)| \le C\,\big|v_{,1}\big|_{\underline{\mathcal{R}}^d_0}$. Recalling that $\widetilde{U^\ell_\epsilon}$ satisfies

$$\forall v \in \mathcal{V}^d_\epsilon, \quad 0 = \sum_{i=1}^{5} a^i_\ell \mathsf{p}^i(\widetilde{U^\ell_\epsilon}, v), \tag{37}$$

setting $w_\ell = \widetilde{U^\ell_\epsilon} - \widehat{U^0_\epsilon}$ and taking $v = w_\ell$ in(37), we get

$$0 = \sum_{i=1}^{5} a^i_\ell \mathsf{p}^i(w_\ell, w_\ell) + \sum_{i=1}^{5} a^i_\ell \mathsf{p}^i(\widehat{U^0_\epsilon}, w_\ell) \tag{38}$$

$$= \sum_{i=1}^{5} a^i_\ell \mathsf{p}^i(w_\ell, w_\ell) + a^3_\ell \mathsf{p}^3(\widehat{U^0_\epsilon}, w_\ell) + \sum_{i=4}^{5}(a^i_\ell - a^i_0)\mathsf{p}^i(\widehat{U^0_\epsilon}, w_\ell) + \mathsf{q}^0_1(w_\ell). \tag{39}$$

Since $a^1_\ell = 1$, $a^2_\ell = O(\ell^{-1})$, $a^3_\ell = O(\ell)$, a^4_ℓ and a^5_ℓ are $O(1)$, $a^4_\ell - a^4_0$ and $a^5_\ell - a^5_0$ are $O(\ell)$ when $\ell \to 0$, using Poincaré's inequality due to the Dirichlet conditions on $D^\pm_\epsilon$ and $\mathcal{I}_L$, we obtain the following estimate

$$\|w_\ell\|^2_{\mathcal{N}^c_\epsilon} + \frac{1}{\ell}\,\big|w_{\ell,1}\big|^2_{\underline{\mathcal{R}}^d_0} + \ell\,\big|w_{\ell,2}\big|^2_{\underline{\mathcal{R}}^d_0} + \|w_\ell\|^2_{\mathcal{R}^L_d} \le C\,\big|w_{\ell,1}\big|_{\underline{\mathcal{R}}^d_0} + C\ell\left(\big|w_{\ell,2}\big|^2_{\underline{\mathcal{R}}^d_0} + \|w_\ell\|_{\mathcal{R}^L_d}\right)$$

where C denotes a positive constant (independent of ℓ). It follows that

$$\|w_\ell\|_{\mathcal{N}^c_\epsilon} \le C\sqrt{\ell}, \quad \big|w_{\ell,1}\big|_{\underline{\mathcal{R}}^d_0} \le C\ell, \quad \big|w_{\ell,2}\big|^2_{\underline{\mathcal{R}}^d_0} \le C, \quad \|w_\ell\|_{\mathcal{R}^L_d} \le C\sqrt{\ell},$$

from which we deduce that $\widetilde{U^\ell_\epsilon}$ converges weakly to $\widehat{U^0_\epsilon}$ in $\hat{\mathcal{V}}^0_\epsilon$.

Moreover (a subsequence of) $\mu\widetilde{U^\ell_\epsilon}_{,1}/\ell$ weakly converges to some σ^*_1 in $L^2(\underline{\mathcal{R}}^d_0)$. Passing to the limit in (37) gives

$$0 = \mathsf{p}^1(\widehat{U^0_\epsilon}, v) + d\int_{\underline{\mathcal{R}}^d_0} \sigma^*_1 v_{,1}\, dx + a^4_0\mathsf{p}^4(\widehat{U^0_\epsilon}, v) + a^5_0\mathsf{p}^5(\widehat{U^0_\epsilon}, v), \quad \forall v \in \mathcal{V}^d_\epsilon.$$

Comparing with (35) and (36), we obtain $d\sigma_1^* = \sigma_1^0$. Hence, all the sequence converges weakly. To obtain the strong convergence, we consider

$$\ell E_\ell := \sum_{i\neq 2} a_\ell^i \mathsf{p}^i(w_\ell, w_\ell) + \frac{\ell}{d}\int_{\underline{\mathcal{R}}_0^d} \mu\Big(\frac{d\widetilde{U_{\epsilon}^{\ell}}_{,1}}{\ell} - \frac{\sigma_1^0}{\mu}\Big)^2 dx.$$

Using (35)–(39), we get

$$\begin{aligned} E_\ell &= \sum_{i=1}^{5} \frac{a_\ell^i}{\ell}\mathsf{p}^i(w_\ell, w_\ell) + 2\mathsf{q}_1^0\left(\frac{w_\ell}{\ell}\right) + \frac{1}{d}\int_{\underline{\mathcal{R}}_0^d} \frac{(\sigma_1^0)^2}{\mu} dx \\ &= -\frac{1}{d}\mathsf{p}^3(\widehat{U_\epsilon^0}, w_\ell) - \sum_{i=4}^{5}\frac{a_\ell^i - a_0^i}{\ell}\mathsf{p}^i(\widehat{U_\epsilon^0}, w_\ell) + \frac{1}{d}\int_{\underline{\mathcal{R}}_0^d} \sigma_1^0\Big(\frac{\sigma_1^0}{\mu} - \frac{d\widetilde{U_{\epsilon}^{\ell}}_{,1}}{\ell}\Big)dx \end{aligned}$$

Since $\lim_{\ell\to 0} E_\ell = 0$, we get that $d\widetilde{U_{\epsilon}^{\ell}}_{,1}/\ell$ converges strongly in $L^2(\underline{\mathcal{R}}_0^d)$ to σ_1^0/μ and $\widetilde{U_\epsilon^\ell}$ converges strongly in $\hat{\mathcal{V}}_\epsilon^0$ to $\widehat{U_\epsilon^0}$. ⊡

<u>Step 4</u> : *Continuity of* P'_ϵ *at* 0.
We start from (26) and write $\mathsf{P}'_\epsilon(\ell) = \mathsf{J}_\epsilon^-(\ell) - \mathsf{J}_\epsilon^+(\ell)$ with

$$\mathsf{J}_\epsilon^-(\ell) = \frac{1}{d}\int_{\underline{\mathcal{R}}_0^d} \frac{\mu}{2}\Big(\big(\widetilde{U_{\epsilon}^{\ell}}_{,2}\big)^2 - \frac{d^2}{\ell^2}\big(\widetilde{U_{\epsilon}^{\ell}}_{,1}\big)^2\Big)dx \tag{40}$$

$$\mathsf{J}_\epsilon^+(\ell) = \frac{1}{L-d}\int_{\mathcal{R}_d^L} \frac{\mu}{2}\Big(\big(\widetilde{U_{\epsilon}^{\ell}}_{,2}\big)^2 - \frac{(L-d)^2}{(L-\ell)^2}\big(\widetilde{U_{\epsilon}^{\ell}}_{,1}\big)^2\Big)dx \tag{41}$$

Passing to the limit when $\ell \to 0$, we get

$$\begin{aligned} \lim_{\ell\to 0} \mathsf{J}_\epsilon^-(\ell) &= \int_{\underline{\mathcal{I}}_0} \frac{\mu}{2}\left((U^0_{\epsilon,2})^2 - (U^0_{\epsilon,1})^2\right) dx_2 = \mathcal{J}_{\underline{\mathcal{I}}_0} \\ \lim_{\ell\to 0} \mathsf{J}_\epsilon^+(\ell) &= \frac{1}{L-d}\int_{\mathcal{R}_d^L} \frac{\mu}{2}\left((\widehat{U_\epsilon^0}_{,2})^2 - \frac{(L-d)^2}{L^2}(\widehat{U_\epsilon^0}_{,1})^2\right) dx \end{aligned}$$

Making the inverse change of variable $\mathbf{x} \mapsto \phi_0^{-1}(\mathbf{x})$ in the integral of the latter equation and using Proposition 8, it follows that the second path-integral is equal to $\mathcal{J}_{\underline{\mathcal{I}}_0}$ like the first one. (There is no discontinuity of the path integral at $l = 0$ because the singularity at $\mathbf{O}$ is weak). Hence $\lim_{\ell\to 0} \mathsf{P}'_\epsilon(\ell) = 0$. ⊡ □

Note that we could also get $\mathsf{P}'_\epsilon(0) = 0$ from the estimates obtained in Step 3. Moreover, we could refine these estimates to obtain the leading term(s) in the expansion of $\mathsf{P}_\epsilon(0) - \mathsf{P}_\epsilon(\ell)$ with respect to ℓ, following the method presented in [6] for the calculation of energy release rates and based on blow-up techniques, but it is beyond the scope of this paper.

3.2.4 Case $0 \le \epsilon < 1$ *and* $\ell = L$

The proof of the regularity of P_ϵ at L is fairly similar to that at 0: we must introduce a new change of variable to compare U_ϵ^L with U_ϵ^ℓ and we benefit from the fact that the singularity disappears when $\ell = L$. Indeed, when $\ell = L$, the boundary conditions become $u_{,2}(x_1, 0) = 0$ and $u(L, x_2) = 0$ near $(L, 0)$, *i.e.* a change of boundary conditions from Dirichlet to Neumann at a corner of angle $\pi/2$. By Grisvard's formula [24], there is no more singularity and $U_\epsilon^L \in H^2(\underline{\mathcal{R}}_l^L)$ when $0 < l < L$. (This is specific to anti-plane elasticity, in plane elasticity the singularity exists, in general, but is weak like at the tip of the notch.) Of course, the presence of the notch has no influence and we have

Proposition 11 $\forall \epsilon \in [0, 1)$, *$\mathsf{P}_\epsilon$ is continuously differentiable at L and $\mathsf{P}'_\epsilon(L) = 0$.*

PROOF. The proof is divided into 2 steps.

Step 1 : *Transport of U_ϵ^L into $\mathcal{U}_\epsilon^d$ and convergence of $\widetilde{U_\epsilon^\ell}$.*

Let $\phi_L : \mathcal{R}_0^L \to \mathcal{R}_0^d, \mathbf{x} \mapsto dx_1/L\mathbf{e}_1 + x_2\mathbf{e}_2$. With $v : \Omega_\epsilon^L \to \mathbb{R}$ is associated $\widehat{v} : \Omega_\epsilon^d \to \mathbb{R}$ by

$$\widehat{v}(\mathbf{x}) = \begin{cases} v(\mathbf{x}) & \text{in} \quad \mathcal{N}_\epsilon^c \\ v \circ {\phi_L}^{-1}(\mathbf{x}) & \text{in} \quad \underline{\mathcal{R}}_0^d \\ 0 & \text{in} \quad \mathcal{R}_d^L \end{cases}, \tag{42}$$

the image of $\mathcal{V}_\epsilon^L$ by this isomorphism is $\widehat{\mathcal{V}_\epsilon^L}$, a (weakly) closed subspace of $\mathcal{V}_\epsilon^d$. We still have $\widehat{U_\epsilon^L} = \overline{U}_\epsilon + \widehat{V_\epsilon^L}$. Inserting the change of variable (42) into (21) leads to

$$\mathsf{P}_\epsilon(L) = \min_{\widehat{v} \in \widehat{\mathcal{V}_\epsilon^L}} \left\{ \frac{1}{2}\mathsf{p}^1(\widehat{v} + \overline{U}_\epsilon, \widehat{v} + \overline{U}_\epsilon) + \frac{1}{2}\sum_{i=2}^{3} a_L^i \mathsf{p}^i(\widehat{v}, \widehat{v}) \right\} \tag{43}$$

where the p^i's are the same as in (24), whereas the a_L^i's are obtained by setting $\ell = L$. The minimizer is $\widehat{V_\epsilon^L}$ and $\widehat{U_\epsilon^L}$ satisfies

$$\forall \widehat{v} \in \widehat{\mathcal{V}_\epsilon^L}, \quad 0 = \sum_{i=1}^{3} a_L^i \mathsf{p}^i(\widehat{U_\epsilon^L}, \widehat{v}). \tag{44}$$

We deduce from (44) that $\widehat{U_\epsilon^L}$ satisfies the "push-forward" equilibrium equations

$$\Delta \widehat{U_\epsilon^L} = 0 \text{ in } \mathcal{N}_\epsilon^c \setminus \mathcal{I}_0, \quad a_L^2 \widehat{U_\epsilon^L}_{,11} + a_L^3 \widehat{U_\epsilon^L}_{,22} = 0 \text{ in } \underline{\mathcal{R}}_0^d$$

and the continuity of the normal stress on $\mathcal{I}_0$ reads now $U_{\epsilon}^L{}_{,1}(0-, x_2) = a_L^2 \widehat{U_\epsilon^L}_{,1}(0+, L)$. The normal stress $\sigma_1^L := \mu\, U_{\epsilon}^L{}_{,1}\Big|_{\mathcal{I}_L}$ on $\mathcal{I}_L$ reads $\sigma_1^L(x_2) = \mu a_L^2 \widehat{U_\epsilon^L}_{,1}(d-, x_2)$. Note that $\sigma_1^L \in L^2(\mathcal{I}_L)$ because there is no singularity at $(L, 0)$.

Multiplying the equilibrium equations by $v \in \mathcal{V}_\epsilon^d$, integrating over $\mathcal{N}_\epsilon^c \cup \underline{\mathcal{R}}_0^d$, integrating by parts and accounting for the boundary conditions lead to

$$\sum_{i=1}^{3} a_L^i \mathsf{p}^i(\widehat{U_\epsilon^L}, v) = \mathsf{q}_1^L(v), \quad \forall v \in \mathcal{V}_\epsilon^d \tag{45}$$

with

$$\mathsf{q}_1^L(v) := -\int_{\mathcal{R}_d^L} \sigma_1^L v_{,1} dx. \tag{46}$$

Recalling that $\widetilde{U_\epsilon^\ell}$ satisfies (37), setting $w_\ell = \widetilde{U_\epsilon^\ell} - \widehat{U_\epsilon^L}$ and taking $v = w_\ell$, we get

$$0 = \sum_{i=1}^{5} a_\ell^i \mathsf{p}^i(w_\ell, w_\ell) + \sum_{i=1}^{3} a_\ell^i \mathsf{p}^i(\widehat{U_\epsilon^L}, w_\ell) \tag{47}$$

$$= \sum_{i=1}^{5} a_\ell^i \mathsf{p}^i(w_\ell, w_\ell) + \sum_{i=2}^{3} (a_\ell^i - a_L^i) \mathsf{p}^i(\widehat{U_\epsilon^L}, w_\ell) + \mathsf{q}_1^L(w_\ell). \tag{48}$$

Note that $a_\ell^1 = 1$, a_ℓ^2 and a_ℓ^3 are $O(1)$, $a_\ell^4 = O((L-\ell)^{-1})$, $a_\ell^5 = O(L-\ell)$, $a_\ell^2 - a_L^2$ and $a_\ell^3 - a_L^3$ are $O(L-\ell)$ when $\ell \to L$. Using Poincaré's inequality due to the Dirichlet conditions on $D_\epsilon^\pm$ and $\mathcal{I}_L$, we obtain the following estimate

$$\|w_\ell\|^2_{\Omega_\epsilon^d \setminus \mathcal{R}_d^L} + \frac{1}{L-\ell}\left|w_{\ell,1}\right|^2_{\mathcal{R}_d^L} + (L-\ell)\left|w_{\ell,2}\right|^2_{\mathcal{R}_d^L} \le C\left|w_{\ell,1}\right|_{\mathcal{R}_d^L} + C(L-\ell)\|w_\ell\|_{\Omega_\epsilon^d \setminus \mathcal{R}_d^L}$$

where C denotes a positive constant (independent of ℓ). It follows that

$$\|w_\ell\|_{\Omega_\epsilon^d \setminus \mathcal{R}_d^L} \le C\sqrt{L-\ell}, \quad \left|w_{\ell,1}\right|_{\mathcal{R}_d^L} \le C(L-\ell), \quad \left|w_{\ell,2}\right|^2_{\mathcal{R}_d^L} \le C,$$

from which we deduce that $\widetilde{U_\epsilon^\ell} - \widehat{U_\epsilon^L}$ converges weakly to 0 in $\widehat{\mathcal{V}_\epsilon^L}$.

Moreover (a subsequence of) $\mu\widetilde{U_{\epsilon}^\ell}_{,1}/(L-\ell)$ weakly converges to some $\sigma_1^\star$ in $L^2(\mathcal{R}_d^L)$. Passing to the limit in (37) gives

$$0 = \sum_{i=1}^{3} a_L^i \mathsf{p}^i(\widehat{U_\epsilon^L}, v) + (L-d)\int_{\mathcal{R}_d^L} \sigma_1^\star v_{,1}\, dx, \quad \forall v \in \mathcal{V}_\epsilon^d.$$

Comparing with (45) and (46), we obtain $(L-d)\sigma_1^\star = \sigma_1^L$. Hence, all the sequence converges weakly. The strong convergence is obtained as in the proof of Proposition 10, step 3, and we get that $(L-d)\widetilde{U_{\epsilon}^\ell}_{,1}/(L-\ell)$ converges strongly to σ_1^L/μ in $L^2(\mathcal{R}_d^L)$, $\widetilde{U_\epsilon^\ell} - \widehat{U_\epsilon^L}$ converges strongly to 0 in $\widehat{\mathcal{V}_\epsilon^L}$ and $(\widehat{U_\epsilon^L} - \widetilde{U_\epsilon^\ell})/\sqrt{L-\ell}$ converges strongly to 0 in $H^1(\Omega_\epsilon^d \setminus \mathcal{R}_d^L)$. ⊡

Step 2 : $\mathsf{P}_\epsilon'(L) = \lim_{\ell\to L} \mathsf{P}_\epsilon'(\ell) = 0$.

To prove that $\lim_{\ell\to L} \mathsf{P}_\epsilon'(\ell) = 0$, we start from $\mathsf{P}_\epsilon'(\ell) = \mathsf{J}_\epsilon^-(\ell) - \mathsf{J}_\epsilon^+(\ell)$ with $\mathsf{J}_\epsilon^\pm(\ell)$ given by (40)–(41). Passing to the limit when $\ell \to L$, we get

$$\lim_{\ell\to L} \mathsf{J}_\epsilon^-(\ell) = \frac{1}{d}\int_{\underline{\mathcal{R}}_0^d} \frac{\mu}{2}\Big(\big(\widehat{U_{\epsilon}^L}_{,2}\big)^2 - \frac{d^2}{L^2}\big(\widehat{U_{\epsilon}^L}_{,1}\big)^2\Big)dx$$

$$\lim_{\ell\to L} \mathsf{J}_\epsilon^+(\ell) = \mathcal{J}_{\mathcal{I}_L} = -\int_{\mathcal{I}_L} \frac{(\sigma_1^L)^2}{2\mu} dx_2$$

Making the inverse change of variable $\mathbf{x} \mapsto \phi_L^{-1}(\mathbf{x})$ in the integral of the former equation above and using Proposition 8, it follows that the first path-integral is equal to $\mathcal{J}_{\mathcal{I}_L}$ like the second one. Hence $\lim_{\ell\to L} \mathsf{P}_\epsilon'(\ell) = 0$.

It remains to prove that $\mathsf{P}_\epsilon'(L) = 0$. Using (23), (37), (43) and (45), we get

$$\begin{aligned}\mathsf{P}_\epsilon(L) - \mathsf{P}_\epsilon(\ell) &= \frac{1}{2}\sum_{i=1}^{3} a_L^i \mathsf{p}^i(\widehat{U_\epsilon^L}, \widehat{U_\epsilon^L}) - \frac{1}{2}\sum_{i=1}^{5} a_\ell^i \mathsf{p}^i(\widetilde{U_\epsilon^\ell}, \widetilde{U_\epsilon^\ell}) \\ &= \frac{1}{2}\sum_{i=2}^{3}(a_L^i - a_\ell^i)\mathsf{p}^i(\widehat{U_\epsilon^L}, \widetilde{U_\epsilon^\ell}) - \frac{1}{2}\mathsf{q}_1^L(\widetilde{U_\epsilon^\ell}).\end{aligned}$$

Dividing by $L-\ell$ and passing to the limit when $\ell \to L$ yields

$$\begin{aligned}\mathsf{P}_\epsilon'(L) &= -\frac{d}{2L^2}\mathsf{p}^2(\widehat{U_\epsilon^L}, \widehat{U_\epsilon^L}) + \frac{1}{2d}\mathsf{p}^3(\widehat{U_\epsilon^L}, \widehat{U_\epsilon^L}) + \int_{\mathcal{I}_L} \frac{(\sigma_1^L)^2}{2\mu} dx_2 \\ &= \lim_{\ell\to L} \mathsf{J}_\epsilon^-(\ell) - \mathcal{J}_{\mathcal{I}_L}\end{aligned}$$

and hence $\mathsf{P}_\epsilon'(L) = 0$. ⊡ □

3.3 Check of the monotonicity of P_ϵ and some conjectures

The strict monotonicity of P_ϵ is obtained via harmonic properties of U_ϵ^ℓ. Specifically, we have

Proposition 12 *For each $\epsilon \in [0,1)$, $\ell \mapsto \mathsf{P}_\epsilon(\ell)$ is decreasing, and, for each $\ell \in [0,L]$, $\epsilon \mapsto \mathsf{P}_\epsilon(\ell)$ is decreasing.*

PROOF. Let $0 \leq l_1 < l_2 \leq L$. Since $\mathcal{U}_\epsilon^{l_2} \supset \mathcal{U}_\epsilon^{l_1}$ and since $\mathsf{P}_\epsilon(\ell)$ is the minimum of the elastic energy over $\mathcal{U}_\epsilon^\ell$ for each ℓ, we have $\mathsf{P}_\epsilon(l_2) \leq \mathsf{P}_\epsilon(l_1)$. Let us prove that the inequality is strict, by contradiction. Assume that $\mathsf{P}_\epsilon(l_2) = \mathsf{P}_\epsilon(l_1)$, then, because of the uniqueness of the minimizer of the elastic energy over $\mathcal{U}_\epsilon^\ell$ for each ℓ, we have $U_\epsilon^{l_1} = U_\epsilon^{l_2}$. Moreover, $U_\epsilon^{l_1}$ is an harmonic function on $\Omega_\epsilon^{l_1}$. Consider the line segment $\Gamma_1^2 = (l_1, l_2) \times \{0\}$. On Γ_1^2, $U_\epsilon^{l_1} = 0$ as the minimizer over $\mathcal{U}_\epsilon^{l_1}$, by symmetry. But, as a minimizer over $\mathcal{U}_\epsilon^{l_2}$, it must also satisfy $(U_\epsilon^{l_1})_{,2} = 0$. Hence $\nabla U_\epsilon^{l_1} = 0$ on Γ_1^2 and the associated holomorphic function $\mathsf{f}_\epsilon^{l_1}$ too, see Remark 3. Therefore $\mathsf{f}_\epsilon^{l_1}$ and $U_\epsilon^{l_1}$ must vanishes on $\Omega_\epsilon^{l_1}$, what it is incompatible with the boundary conditions. Hence, $\mathsf{P}_\epsilon(l_2) \neq \mathsf{P}_\epsilon(l_1)$.

Let $\ell \in [0,L]$ and $0 \leq \epsilon_1 < \epsilon_2 < 1$. Let $U_{\epsilon_1}^\ell$ be the minimizer of $\int_{\Omega_{\epsilon_1}^\ell} \frac{\mu}{2} \nabla u \cdot \nabla u \, dx$ over $\mathcal{U}_{\epsilon_1}^\ell$. Its restriction to $\Omega_{\epsilon_2}^\ell$ belongs to $\mathcal{U}_{\epsilon_2}^\ell$. Since $U_{\epsilon_1}^\ell$ is harmonic, it is not constant in $\mathcal{N}_{\epsilon_2} \setminus \mathcal{N}_{\epsilon_1}$. Therefore

$$\mathsf{P}_{\epsilon_1}(\ell) = \int_{\Omega_{\epsilon_1}^\ell} \frac{\mu}{2} \nabla U_{\epsilon_1}^\ell \cdot \nabla U_{\epsilon_1}^\ell \, dx > \int_{\Omega_{\epsilon_2}^\ell} \frac{\mu}{2} \nabla U_{\epsilon_1}^\ell \cdot \nabla U_{\epsilon_1}^\ell \, dx \geq \mathsf{P}_{\epsilon_2}(\ell).$$

□

All the above properties of monotonicity or regularity of P_ϵ and its derivatives might remain true in a more general setting because their proofs do not actually rest on the particular data of the loading and the geometry. Other properties, like the positivity of $-\mathsf{P}_0'(0)$ or the positivity of $-\mathsf{P}_\epsilon'$ in $(0,L)$, are expected. However, at the present time, we are not able to derive them by analytic arguments and must check them by finite element computations. As regards convexity properties of P_ϵ, the situation is quite different because such properties are strongly dependent on the geometry, the type of loading and even on the locus where the displacements or the forces are applied. In our particular case, for reasons invoked below, we can expect some convexity properties like the strict convexity of P_0. But, their check will be also numerical. Consequently, we set all the additional properties needed for the sequel as the following conjecture

Conjecture 1 P_0 *is strictly convex and* $\mathsf{P}_\epsilon' \geq \mathsf{P}_0'$, $\forall \epsilon \in [0,1)$.

Note that the strict convexity of P_0 implies that $\mathsf{P}_0'(0) < 0$ because P_0 is decreasing. Let us give some arguments in favor of this conjecture.

1. If the body is slender, *i.e.* if the ratio H/L is small, we can construct fairly good approximations of $\mathsf{P}_0(\ell)$ for ℓ enough far from 0 and L with the help of asymptotic methods. Indeed, it is easy to prove that $\mu H/\ell$ is the leading term of the expansion of $\mathsf{P}_0(\ell)$ when H/L goes to 0. Since $\mu H/\ell$ is strictly convex, we can expect that so is P_0 at least enough far from 0 and L. Since $\mathsf{P}_0'(L) = 0$, P_0 is certainly strictly convex near $\ell = L$.
2. We know by Proposition 12 that $\mathsf{P}_\epsilon \leq \mathsf{P}_0$, *i.e.* the smaller the amount of matter, the smaller the elastic energy. We can expect that the same type of inequality holds for the energy release rates, but there is no variational arguments available at the present time.

3.4 Check of the regularity of $\mathsf{P}_\epsilon(\ell)$ with respect to ϵ

For a given $\ell \in [0, L]$, to study the dependence of $\mathsf{P}_\epsilon(\ell)$ on ϵ, we make once more a change of variables which send the ϵ-dependent domain Ω_ϵ onto the fix domain Ω_0. Specifically, let ϕ_ϵ be the following lipschitzian homeomorphism from Ω_ϵ onto Ω_0

$$\phi_\epsilon(\mathbf{x}) = \begin{cases} x_1\mathbf{e}_1 + \frac{1}{1-\epsilon}(x_2 + \text{sign}(x_2)\epsilon x_1)\mathbf{e}_2 & \text{if } \mathbf{x} \in \mathcal{N}_1 \setminus \mathcal{N}_\epsilon \\ \mathbf{x} & \text{otherwise} \end{cases}. \tag{49}$$

With each field v defined on Ω_ϵ^ℓ is associated its push-forward $\widetilde{v} = v \circ \phi_\epsilon^{-1}$ defined on Ω_0^ℓ. Inserting this change of variable into (21) leads to

$$\mathsf{P}_\epsilon(\ell) = \min_{\widetilde{v}\in\mathcal{V}_0^\ell} \left\{ \frac{1}{2}\sum_{i=1}^{4} a_\epsilon^i \mathsf{p}_i(\widetilde{v}, \widetilde{v}) + \sum_{i=1}^{2} b_\epsilon^i \mathsf{q}_i(\widetilde{v}) + c_\epsilon \right\} \tag{50}$$

with

$$\begin{cases} a_\epsilon^1 = 1 & , \quad \mathsf{p}_1(u,v) = \int_{\Omega_0^\ell\setminus\mathcal{N}_1} \mu\nabla u\cdot\nabla v\ dx \\ a_\epsilon^2 = 1-\epsilon & , \quad \mathsf{p}_2(u,v) = \int_{\mathcal{N}_1\setminus\mathcal{N}_0} \mu u_{,1}v_{,1}\ dx \\ a_\epsilon^3 = \frac{1+\epsilon^2}{1-\epsilon} & , \quad \mathsf{p}_3(u,v) = \int_{\mathcal{N}_1\setminus\mathcal{N}_0} \mu u_{,2}v_{,2}\ dx \\ a_\epsilon^4 = \epsilon & , \quad \mathsf{p}_4(u,v) = \int_{\mathcal{N}_1\setminus\mathcal{N}_0} \text{sign}(x_2)\mu(u_{,1}v_{,2}+u_{,2}v_{,1})\ dx \\ b_\epsilon^1 = 1-\epsilon & , \quad \mathsf{q}_1(v) = -\int_{\mathcal{N}_1\setminus\mathcal{N}_0} \text{sign}(x_2)\frac{\mu}{H}v_{,1}\ dx \\ b_\epsilon^2 = 1 & , \quad \mathsf{q}_2(v) = -\int_{\mathcal{N}_1^c} \text{sign}(x_2)\frac{\mu}{H}v_{,1}\ dx \\ c_\epsilon = (1-\frac{\epsilon}{2})\mu \end{cases} \tag{51}$$

The minimizer in (50) is $\widetilde{V_\epsilon^\ell}$, the push-forward of V_ϵ^ℓ, and $\widetilde{V_\epsilon^\ell} = \widetilde{U_\epsilon^\ell} - \widetilde{\overline{U}_\epsilon}$. Owing to the Lemma 18, we get

Proposition 13 *For each $\ell \in [0, L]$, $\epsilon \mapsto \mathsf{P}_\epsilon(\ell)$ is indefinitely differentiable on $[0, 1)$. Moreover, P_ϵ converges uniformly to P_0 when ϵ goes to 0, whereas P'_ϵ converges uniformly to P'_0 on any compact $[l, L]$ such that $l > 0$.*

Proof. The four a_ϵ^i, the two b_ϵ^i and c_ϵ are indefinitely differentiable on $[0, 1)$, $\sum_{i=1}^4 a_\epsilon^i \mathsf{p}_i$ is uniformly coercive on $\mathcal{V}_0^\ell$ in any closed sub-interval of $[0, 1)$, the p_i's and the q_i's are continuous on $\mathcal{V}_0^\ell$. Hence, we can use Lemma 18 and Remark 6 of the Appendix with $\lambda = \epsilon$, $\Lambda = [0, 1)$ and $\mathcal{H} = \mathcal{V}_0^\ell$ equipped with the norm of $H^1(\Omega_0^\ell)$ to obtain that $\epsilon \mapsto \mathsf{P}_\epsilon(\ell)$ is indefinitely differentiable on $[0, 1)$.

Let us prove now the uniform convergence of P_ϵ to P_0 and more precisely that there exists $C > 0$ such that $\forall \ell \in [0, L]$ and ϵ small enough, $|\mathsf{P}_\epsilon(\ell) - \mathsf{P}_0(\ell)| \le C\epsilon$. Let $\ell \in [0, L]$. Taking $\widetilde{v} = 0$ in (50), we get $\mathsf{P}_\epsilon(\ell) \le c_\epsilon \le \mu$ and hence $\left|\nabla\widetilde{U_\epsilon^\ell}\right|_{\Omega_\epsilon^\ell} \le \sqrt{2}$. We easily check that there exists $\alpha > 0$ such that $\sum_{i=1}^4 a_\epsilon^i \mathsf{p}_i(u, u) \ge \alpha\, |\nabla u|_{\Omega_\epsilon^\ell}^2$ for ϵ small enough and for all $u \in \mathcal{U}_0^\ell$. (Remark that only p^1 depends on ℓ.) By (62), we get

$$\sum_{i=1}^{4} a_\epsilon^i \mathsf{p}_i(v_\epsilon^\ell, v_\epsilon^\ell) + \sum_{i=1}^{4} \frac{a_\epsilon^i - a_0^i}{\epsilon} \mathsf{p}_i(\widetilde{U_\epsilon^\ell}, v_\epsilon^\ell) = 0$$

where $v_\epsilon^\ell = (\widetilde{U_\epsilon^\ell} - U_0^\ell)/\epsilon$. Hence, $\left|\nabla v_\epsilon^\ell\right|_{\Omega_\epsilon^\ell} \leq C$. Since

$$\begin{aligned}\mathsf{P}_\epsilon(\ell) - \mathsf{P}_0(\ell) &= \sum_{i=1}^4 a_\epsilon^i \mathsf{p}_i(U_0^\ell + \epsilon v_\epsilon^\ell, U_0^\ell + \epsilon v_\epsilon^\ell) - \sum_{i=1}^4 a_0^i \mathsf{p}_i(U_0^\ell, U_0^\ell)\\ &= \sum_{i=1}^4 (a_\epsilon^i - a_0^i)\mathsf{p}_i(U_0^\ell, U_0^\ell) + 2\epsilon \sum_{i=1}^4 a_\epsilon^i \mathsf{p}_i(U_0^\ell, v_\epsilon^\ell) + \epsilon^2 \sum_{i=1}^4 a_\epsilon^i \mathsf{p}_i(v_\epsilon^\ell, v_\epsilon^\ell),\end{aligned}$$

we get $|\mathsf{P}_\epsilon(\ell) - \mathsf{P}_0(\ell)| \leq C\epsilon$.

For the convergence of P_ϵ', we start from Proposition 7 and (25), and note that $\widetilde{U_\epsilon^\ell} = U_\epsilon^\ell$ in $\mathcal{R}_0^L$. Using $\left|\nabla v_\epsilon^\ell\right|_{\Omega_\epsilon^\ell} \leq C$ and comparing (25) with (29) we easily deduce the uniform convergence on any $[l_1, l_2]$ with $0 < l_1 < l_2 < L$. Its extension up to L needs to use the special transformation (42) and the estimates obtained in Proposition 11. The proof is left to the reader. □

Remark 5 *Note that there is no singular behavior of P_ϵ at $\epsilon = 0$. At given ℓ, the transformation of a crack into a notch is a regular perturbation, at least as long as one considers only the energy but not its derivative with respect to ℓ.*

3.5 Onset and evolution of the crack

The surface energy is independent of ϵ and simply reads $\mathsf{S}(\ell) = \mathsf{G}_c\ell$. Therefore, if one chose the units in such a manner that $\mathsf{G}_c = 1$, the maps $\mathsf{P}_\epsilon \circ \mathsf{S}^{-1}$ and $\mathsf{g}_\epsilon \circ \mathsf{S}^{-1}$ could be identified with P_ϵ and $\mathsf{G}_\epsilon := -\mathsf{P}_\epsilon'$, respectively.

We establish in this subsection the main properties of the crack evolution which depend on whether we consider the *G-law* or the *FM-law*, and on whether $\epsilon = 0$ or $0 < \epsilon < 1$. Moreover, we will distinguish, among these properties, the ones which do not use the conjecture.

Proposition 14 *For a genuine notch,* i.e. *when $0 < \epsilon < 1$, the unique solution of the* G-law *is $\ell^\epsilon(t) = 0$ for all $t \geq 0$.*

On the other hand, each solution $t \mapsto \ell^\epsilon(t)$ of the FM-law *necessarily enjoys the following properties:*

1. *There exists $\mathsf{t}_\mathsf{i}^\epsilon > 0$ and $\ell_\mathsf{i}^\epsilon \in (0, L)$ such that $\ell^\epsilon(t) = 0$ for $t < \mathsf{t}_\mathsf{i}^\epsilon$ and $\ell^\epsilon(\mathsf{t}_\mathsf{i}^\epsilon+) = \ell_\mathsf{i}^\epsilon$;*
2. *$\mathsf{t}_\mathsf{i}^\epsilon$ and ℓ_i^ϵ satisfy*

$$\mathsf{P}_\epsilon(\ell_\mathsf{i}^\epsilon) - \mathsf{P}_\epsilon(0) = \ell_\mathsf{i}^\epsilon \mathsf{P}_\epsilon'(\ell_\mathsf{i}^\epsilon), \qquad \mathsf{t}_\mathsf{i}^\epsilon = \sqrt{\frac{\mathsf{G}_c}{-\mathsf{P}_\epsilon'(\ell_\mathsf{i}^\epsilon)}} \tag{52}$$

3. *The crack cannot reach the end L in a finite time, but $\lim_{t\to\infty} \ell^\epsilon(t) = L$.*

Proof. Since $\mathsf{P}_\epsilon'(0) = 0$, the first part of the Proposition is a direct consequence of Proposition 1 and $\ell^\epsilon = 0$ is the unique solution of the *G-law*. For the second part, we can use Proposition 5. Since $\mathsf{P}_\epsilon'(L) = 0$, we are in the case 2 and L cannot be reached at a finite time. □

When $\epsilon = 0$, if we adopt the Conjecture 1, we can use Propositions 2 and 4 to obtain

Proposition 15 *Under the Conjecture 1, when $\epsilon = 0$, the* G-law *and the* FM-law *admit the same and unique solution given by*

$$\ell^0(t) = \begin{cases} 0 & \text{if} \quad t \le \mathsf{t}_\mathsf{i}^0 = \sqrt{\frac{\mathsf{G}_c}{-\mathsf{P}_0'(0)}} \\ (\mathsf{P}_0')^{-1}(-\mathsf{G}_c/t^2) & \text{if} \quad t \ge \mathsf{t}_\mathsf{i}^0 \end{cases} \tag{53}$$

Note again that L cannot be reached at a finite t because $\mathsf{P}_0'(L) = 0$.

Comparing the two preceding Propositions, we immediately see that the evolution predicted by the *G-law* is not continuous with respect to the parameter ϵ. On the other hand, the evolution given by the *FM-law* is continuous with respect to ϵ as it is shown in the following Proposition. Note that the first part of the continuity property is established without having recourse to the conjecture.

Proposition 16 *At each $t \ge 0$, a solution $\ell^\epsilon(t)$ of the* FM-law *corresponding to $\epsilon > 0$ converges pointwise to a solution $\ell^0(t)$ of the* FM-law *corresponding to $\epsilon = 0$ when ϵ goes to 0. Furthermore, under the Conjecture 1, the initiation length ℓ_i^ϵ converges to 0 and the initiation loading $\mathsf{t}_\mathsf{i}^\epsilon$ converges to t_i^0 when ϵ goes to 0.*

Proof. Fix $t \ge 0$. For $\epsilon \in [0,1)$, a solution $\ell^\epsilon(t)$ of the *FM-law* is a minimizer of the energy at t, *i.e.*

$$t^2\mathsf{P}_\epsilon(\ell^\epsilon(t)) + \mathsf{G}_c\ell^\epsilon(t) \le t^2\mathsf{P}_\epsilon(l) + \mathsf{G}_c l, \quad \forall l \in [0, L]. \tag{54}$$

When ϵ goes to 0, since $\ell^\epsilon(t)$ is bounded, there exists a subsequence (still denoted by $\ell^\epsilon(t)$) which converges to some $\ell^0(t)$. By virtue of Proposition 13, we can pass to the limit in the optimality condition above and obtain

$$t^2\mathsf{P}_0(\ell^0(t)) + \mathsf{G}_c\ell^0(t) \le t^2\mathsf{P}_0(l) + \mathsf{G}_c l, \quad \forall l \in [0, L].$$

Hence, $\ell^0(t)$ is a solution of the *FM-law* corresponding to $\epsilon = 0$. By lack of uniqueness, we cannot say more without invoking the conjecture. Under the conjecture 1, since the response is unique when $\epsilon = 0$, all the sequence $\ell^\epsilon(t)$ converges to $\ell^0(t)$ given by (53).

Moreover, the sequence ℓ_i^ϵ is bounded, we can extract a subsequence (still denoted ℓ_i^ϵ) converging to some ℓ_i^0. If $\ell_i^0 \ne 0$, we can use Proposition 13 and, passing to the limit in $(52)_1$, we get

$$\mathsf{P}_0(\ell_i^0) - \mathsf{P}_0(0) = \ell_i^0\mathsf{P}_0'(\ell_i^0).$$

But since P_0 is supposed strictly convex, the unique solution is $\ell_i^0 = 0$. Hence all the sequence ℓ_i^ϵ converges to 0. Since $\mathsf{t}_\mathsf{i}^\epsilon \le \sqrt{\mathsf{G}_c\mathsf{S}(L)/(\mathsf{P}_\epsilon(0) - \mathsf{P}_\epsilon(L))}$, the sequence $\mathsf{t}_\mathsf{i}^\epsilon$ is bounded. Extracting a subsequence converging to, say, t_i^*, setting $t = \mathsf{t}_\mathsf{i}^\epsilon$ in (54) and passing to the limit, we get

$$(\mathsf{t}_\mathsf{i}^*)^2\mathsf{P}_0(0) \le (\mathsf{t}_\mathsf{i}^*)^2\mathsf{P}_0(l) + \mathsf{G}_c l, \quad \forall l \in [0, L].$$

Hence $\mathsf{t}_\mathsf{i}^* \le \mathsf{t}_\mathsf{i}^0$. Now, by conjecture 1, we have $\mathsf{t}_\mathsf{i}^\epsilon \ge \sqrt{\mathsf{G}_c/(-\mathsf{P}_0'(\ell_\mathsf{i}^\epsilon))}$. Passing to the limit when ϵ goes to 0, we get $\mathsf{t}_\mathsf{i}^* \ge \mathsf{t}_\mathsf{i}^0$. Hence $\mathsf{t}_\mathsf{i}^* = \mathsf{t}_\mathsf{i}^0$ and all the sequence $\mathsf{t}_\mathsf{i}^\epsilon$ converges to t_i^0. Note that we deduce, from the convergence of $\mathsf{t}_\mathsf{i}^\epsilon$ to t_i^0, the convergence of $\mathsf{P}_\epsilon'(\ell_\mathsf{i}^\epsilon)$ to $\mathsf{P}_0'(0)$ and hence we have $\lim_{\epsilon\downarrow 0}\mathsf{P}_\epsilon'(\ell_\mathsf{i}^\epsilon) = \mathsf{P}_0'(0) < 0 = \lim_{\epsilon\downarrow 0}\mathsf{P}_\epsilon'(0)$. □

The last property will concern the concept of barrier of energy. Let $\epsilon \in (0,1)$ and let us set $\mathcal{E}_\epsilon(t,l) = t^2\mathsf{P}_\epsilon(l) + \mathsf{G}_c l$. Note first that $l = 0$ is a strict local minimum of $\mathcal{E}(t,\cdot)$ for all t, because $\mathsf{P}_\epsilon'(0) = 0$. Indeed, for every t, $\partial\mathcal{E}_\epsilon(t,0)/\partial\ell = \mathsf{G}_c > 0$ and hence there exists

$l_t > 0$ such that $\mathcal{E}_\epsilon(t,0) < \mathcal{E}_\epsilon(t,l)$ for all $l \in (0,l_t)$. At time $\mathsf{t}_\mathsf{i}^\epsilon$, when a crack of length ℓ_i^ϵ appears, both $l=0$ and $l=\ell_\mathsf{i}^\epsilon$ are global minimizers, $\mathsf{E}_\epsilon(\mathsf{t}_\mathsf{i}^\epsilon) := \mathcal{E}_\epsilon(\mathsf{t}_\mathsf{i}^\epsilon,0) = \mathcal{E}_\epsilon(\mathsf{t}_\mathsf{i}^\epsilon,\ell_\mathsf{i}^\epsilon)$ and we have $\mathcal{E}(t,l) \geq \mathsf{E}_\epsilon(\mathsf{t}_\mathsf{i}^\epsilon)$ for all $l \in [0,\ell_\mathsf{i}^\epsilon]$. Since the inequality is strict in a part of the interval, the body must cross a barrier of energy to jump to the new global minimizer. We can define this barrier as

$$\mathcal{B}_\epsilon = \max_{l\in[0,\ell_\mathsf{i}^\epsilon]}\{\mathcal{E}_\epsilon(\mathsf{t}_\mathsf{i}^\epsilon,l) - \mathsf{E}_\epsilon(\mathsf{t}_\mathsf{i}^\epsilon)\}. \tag{55}$$

Since $\mathcal{B}_\epsilon > 0$, a maximiser $\ell_i^{\epsilon *}$ is necessarily in the open interval $(0,\ell_\mathsf{i}^\epsilon)$ and hence such that $\partial\mathcal{E}_\epsilon(\mathsf{t}_\mathsf{i}^\epsilon,\ell_i^{\epsilon *})/\partial\ell = 0$. Therefore $\ell_i^{\epsilon *}$ is such that $\mathsf{P}_\epsilon'(\ell_i^{\epsilon *}) = \mathsf{P}_\epsilon'(\ell_\mathsf{i}^\epsilon)$, see Proposition 6 and Figures 1 and 7. The energy barrier can then be read

$$\mathcal{B}_\epsilon = \mathsf{G}_c\ell_i^{\epsilon *} - (\mathsf{t}_\mathsf{i}^\epsilon)^2(\mathsf{P}_\epsilon(0) - \mathsf{P}_\epsilon(\ell_i^{\epsilon *})). \tag{56}$$

On the other hand, under the conjecture 1, the crack evolves continuously with t when $\epsilon = 0$ and there is no barrier of energy. Thus, we can expect that the continuity property remains true for the barrier of energy and that the barrier will progressively disappear for small ϵ. It is confirmed by the following

Proposition 17 *Under the conjecture 1, the barrier of energy $\mathcal{B}_\epsilon$ that the body must cross when the crack initiates tends to 0 when ϵ tends to 0.*

Proof. We know by Proposition 16 that $\mathsf{t}_\mathsf{i}^\epsilon$ tends to t_i^0 and that ℓ_i^ϵ tends to 0 when $\epsilon \to 0$. Since $0 < \ell_i^{\epsilon *} < \ell_\mathsf{i}^\epsilon$, $\lim_{\epsilon\to 0}\ell_i^{\epsilon *} = 0$. Using Proposition 13, we can pass to the limit in (56) and obtain $\lim_{\epsilon\to 0}\mathcal{B}_\epsilon = 0$. (It is even easy to show that $\mathcal{B}_\epsilon \leq C\ell_i^{\epsilon *}$). □

4 Numerical computations

4.1 Numerical procedure to compute $\mathsf{P}_\epsilon(\ell)$ and $\mathsf{G}_\epsilon(\ell)$

All the computations are made with the finite element method and the industrial code COMSOL. They are made after introducing dimensionless quantities. The dimensions of the body are $H = 1$ and $L = 5$, the shear modulus $\mu = 1$ and $\mathsf{G}_c = 1$. For a given value of $\ell \in (0,L)$ and a given value of $\epsilon \in [0,1)$, we use the symmetry of the body and of the load to mesh only its upper half and prescribe $u = 0$ on the segment $\ell \leq x_1 \leq L$, $x_2 = 0$. We use 6-nodes triangular elements, *i.e.* quadratic Lagrange interpolations. The mesh is refined near the singular corners and a typical mesh contains 25000 elements and 50000 degrees of freedom. We compute the discretized solution (still denoted) U_ϵ^ℓ by solving the linear system. Then, the energy $\mathsf{P}_\epsilon(\ell)$ and the energy release rate $\mathsf{G}_\epsilon(\ell) := -\mathsf{P}_\epsilon'(\ell)$ are obtained by a post-treatment. The energy is simply obtained by a direct integration of the elastic energy density over the body. The derivative of the energy is obtained by using the formula (25), which needs to integrate the different parts of the elastic energy density over the two rectangles $(0,\ell)\times(0,H)$ and $(\ell,L)\times(0,H)$. (The cases $\ell = 0$ and $\ell = L$ are treated by using specific meshes and we compute only U_ϵ^ℓ and $\mathsf{P}_\epsilon(\ell)$.)

For a given ϵ, we compute $\mathsf{P}_\epsilon(\ell)$ and $\mathsf{G}_\epsilon(\ell)$ for ℓ varying from 0 to 5, first by steps of 0.001 in the interval $(0,0.05)$, then by steps of 0.002 in the interval $(0.05,0.2)$, finally by steps of 0.01 in the interval $(0.2,5)$. The computations can be considered as sufficiently accurate for $\ell > 0.002$. Below this value, if we try to refine the mesh near the corner of the notch, the results become mesh-sensitive, the linear system becomes bad-conditioned. Since all the interesting part of the graph of G_ϵ is that close to $\ell = 0$ when ϵ is small, we cannot obtain accurate results when ϵ is too small. (Of course, this remark does not apply when $\epsilon = 0$, because $\ell = 0$ is no more a "singular" case and we can use the formula (29).) Consequently, we have only considered values of ϵ larger than 0.02.

4.2 Numerical check of the conjecture

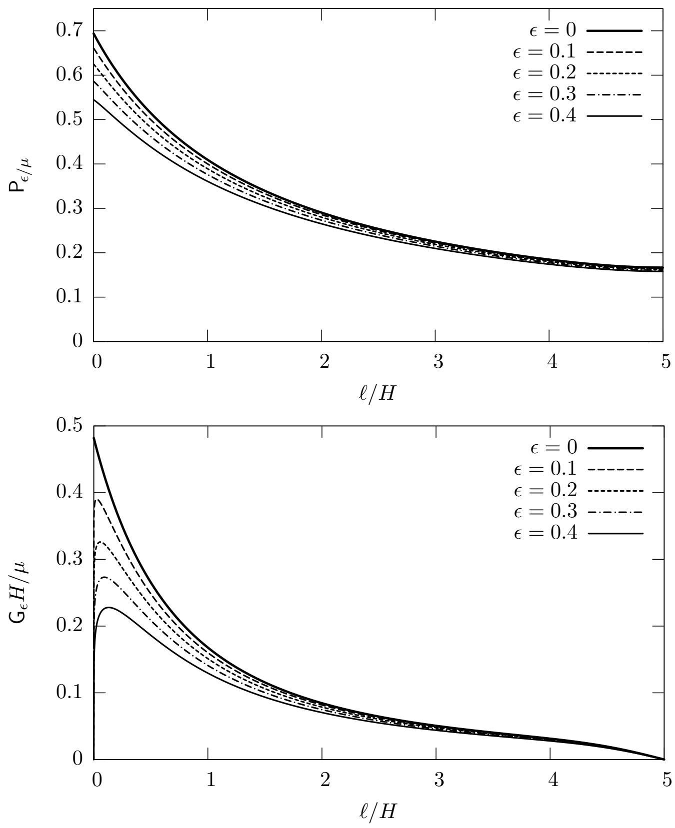

Fig. 3 Top: Graph of $\ell \mapsto \mathsf{P}_\epsilon(\ell)$ on the full range of ℓ for five values of ϵ; Bottom: Graph of $\ell \mapsto \mathsf{G}_\epsilon(\ell)$ on the full range of ℓ for five values of ϵ

In Figure 3 is plotted the graph of P_0 and G_0. As expected in the first part of the conjecture, it appears that G_0 is decreasing and hence that P_0 is strictly convex. In the same Figure are plotted the graphs of P_ϵ and G_ϵ for $\epsilon = 0.1$, 0.2, 0.3 and 0.4. It appears in the graph of P_ϵ that the influence of ϵ is small and essentially visible for small values of ℓ, which is conform to the convergence result of Proposition 13. On the other hand, we know that P_ϵ is not convex since $\mathsf{P}_\epsilon'(0) = 0$. This loss of convexity is absolutely impossible to detect in the graph of P_ϵ. It becomes visible in the graph of G_ϵ where we see that G_ϵ starts from 0 at $\ell = 0$, then increases up to a maximal value at $\ell = \ell_c^\epsilon$, then decreases to 0 when ℓ increases from ℓ_c^ϵ to L. This property has been checked for all tested values of ϵ. Consequently, we are exactly in the situation of Proposition 6. Note that the graph of P_ϵ is below that of P_0, in agreement with Proposition 12, but also that the graph of G_ϵ is below that of G_0, which corresponds to the second part of the conjecture. Figure 4 shows the influence of the angle of the

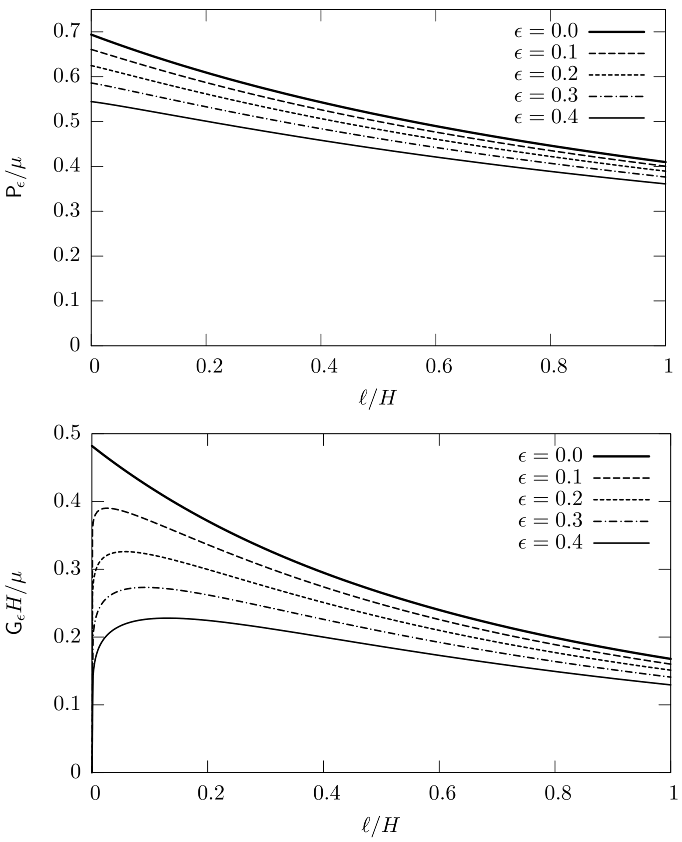

Fig. 4 Top: Graph of P_ϵ near $\ell = 0$ for the five values of ϵ; Bottom: Graph of G_ϵ near $\ell = 0$ for the five values of ϵ

notch on the energy and on the energy release rate for small cracks. We can note the monotony of the graphs with respect to ϵ, which confirms also the conjecture. It enables us to visualize the convergence result of P_ϵ to P_0 and of G_ϵ to G_0 proved in Proposition 13. In particular, we can see that G_ϵ is rapidly increasing near $\ell = 0$, that the maximum converges progressively to $\mathsf{G}_0(0) = 0.482$ and that ℓ_c^ϵ progressively decreases to 0. It is this singular behaviour near 0 which renders the computations less and less accurate for small ℓ when ϵ goes to 0.

4.3 Computed crack evolutions satisfying the *FM-law*

As we have remarked is the preceding subsection, we are in the situation of Proposition 6. For $\epsilon \in (0, L)$, a crack of length $\ell_{\mathsf{i}}^\epsilon$ initiates at $t = \mathsf{t}_{\mathsf{i}}^\epsilon$, then the evolution of the crack satisfies both the *G-law* and the *FM-law*, since G_ϵ is decreasing. The computation of the length of initiation $\ell_{\mathsf{i}}^\epsilon$ requires to solve the equation for ℓ

$$0 = \mathsf{P}_\epsilon(0) - \mathsf{P}_\epsilon(\ell) - \ell \mathsf{G}_\epsilon(\ell). \tag{57}$$

This is achieved by dichotomy, using the fact that the right-hand side of (57) is positive when $\ell > \ell_{\mathsf{i}}^\epsilon$ and that $\ell_{\mathsf{i}}^\epsilon > \ell_c^\epsilon$. For each tested value of ℓ, a new mesh is created, $\mathsf{P}_\epsilon(\ell)$ and $\mathsf{G}_\epsilon(\ell)$ are computed as explained above. The test of convergence is that the absolute value of the right-hand side of (57) must be less than 10^{-6}. The value of $\ell_{\mathsf{i}}^\epsilon$ has been computed by this procedure for ϵ varying from 0.04 to 0.3 by steps of 0.02. The value of $\mathsf{t}_{\mathsf{i}}^\epsilon$ is obtained by (52). For $t > \mathsf{t}_{\mathsf{i}}^\epsilon$, since $t \mapsto \ell^\epsilon(t)$ is increasing, we compute its inverse, that is we compute t for a given $\ell \in (\ell_{\mathsf{i}}^\epsilon, L)$ by using the *G-law* property $t = \sqrt{\mathsf{G}_c/\mathsf{G}_\epsilon(\ell)}$. The computed values of $\ell_{\mathsf{i}}^\epsilon$ are plotted in Figure 5. We see that the dependence of $\ell_{\mathsf{i}}^\epsilon$

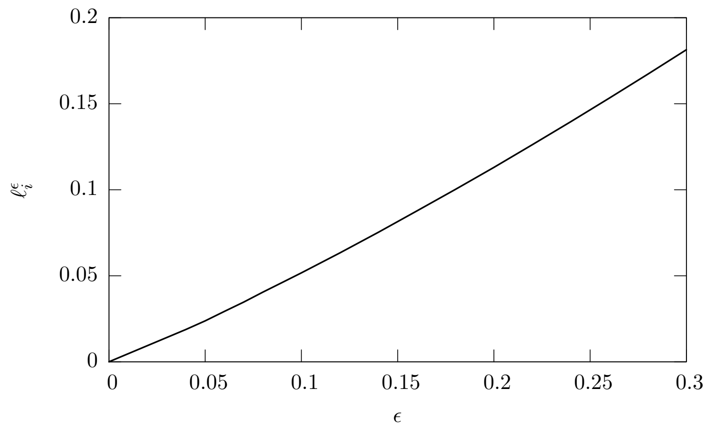

Fig. 5 Computed values of $\ell_{\mathsf{i}}^\epsilon$ for different values of ϵ

on ϵ is almost linear with a slope close to 0.5, *i.e.* $\ell_{\mathsf{i}}^\epsilon \approx \epsilon H/2$.

The evolution of the length of the crack with t is plotted in Figure 6 for different values of ϵ. For $\epsilon = 0$, the evolution is smooth while for $\epsilon \neq 0$ we see the discontinuity at $t = \mathsf{t}_{\mathsf{i}}^\epsilon$. But at a large scale (see the top figure), this discontinuity is practically invisible for small values of ϵ and the response is almost the same as that for $\epsilon = 0$. That illustrates the continuity property of the *FM-law*. At a small scale (see the bottom figure), the discontinuity becomes visible, but the smaller ϵ the smaller the discontinuity.

The difference between the *G-law* and the *FM-law* can be seen on the evolution of the graph of $\ell \mapsto (\mathcal{E}_\epsilon(t, \ell) - \mathcal{E}_\epsilon(t, 0))$ with t. In Figure 7 is plotted the evolution of the graph of this energy difference for $\epsilon = 0.2$. We can see that $\ell = 0$ is always a local minimum and that the slope at $\ell = 0$ is always equal to G_c. When t is close to 0, $\ell = 0$ is the unique local minimizer (and hence the global minimizer). That remains true as long

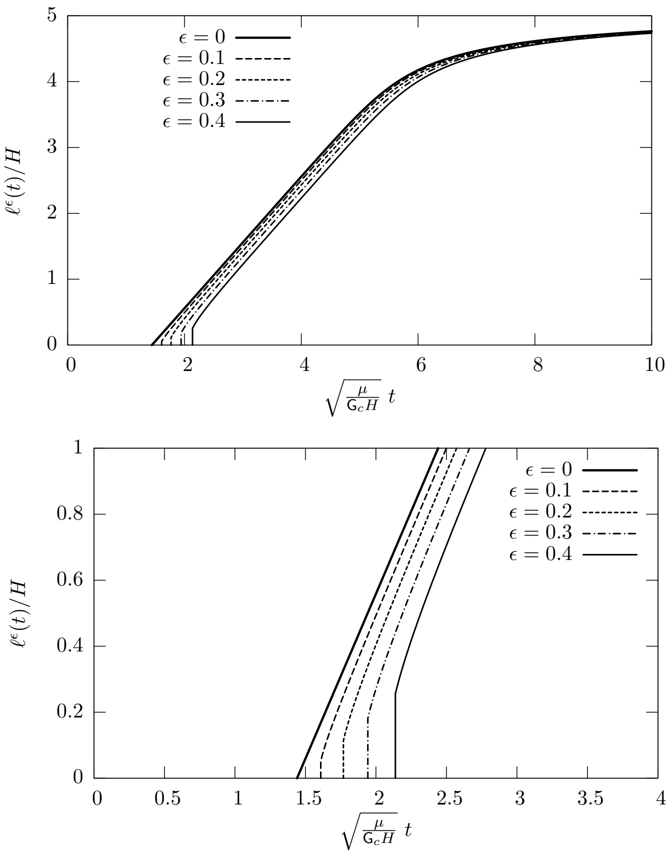

Fig. 6 Top: Graph of $t \mapsto \ell^\epsilon(t)$ on the full range of ℓ for five values of ϵ; Bottom: Graph of $t \mapsto \ell^\epsilon(t)$ at the beginning of the process for the five values of ϵ

as $t \leq t_c^\epsilon$. At $t = t_c^\epsilon$, a second local minimum appears but remains higher than that at 0 as long as $t < \mathsf{t}_\mathsf{i}^\epsilon$. Therefore, because of the energy balance, the crack cannot initiate. At $t = \mathsf{t}_\mathsf{i}^\epsilon$, the two local minima are at the same level and hence are both global minima. Therefore, the crack can initiate, but necessarily by jumping from 0 to ℓ_i^ϵ. As soon as $t > \mathsf{t}_\mathsf{i}^\epsilon$, this second local minimum is below than that at 0 (and becomes the unique global minimum). Therefore, because of the second item of *FM-law*, the crack length must corresponds to this minimizer. The barrier of energy is visible. It corresponds to the increment of energy at $\ell = \ell_i^{\epsilon *}$ when $t = \mathsf{t}_\mathsf{i}^\epsilon$. For $\epsilon = 0.2$, $\ell_\mathsf{i}^\epsilon/H = 0.113$ and the relative energy barrier $\mathcal{B}_\epsilon/\mathcal{P}(\mathsf{t}_\mathsf{i}^\epsilon, 0) = 6.2\ 10^{-4}$.

By comparison, when $\epsilon = 0$, the evolution of the graph of $\ell \mapsto (\mathcal{E}_0(t,\ell) - \mathcal{E}_0(t,0))$ with t is completely different, because of the strict convexity of P_0, see Figure 8. We can see that $\ell = 0$ is both the unique local minimizer and the global minimizer as long as $t \leq \mathsf{t}_\mathsf{i}^0$. As soon as $t > \mathsf{t}_\mathsf{i}^0$, $\ell = 0$ is no more a local minimizer, but a new unique local (and hence global) minimizer appears. At the beginning, it is close to 0, then it increases progressively.

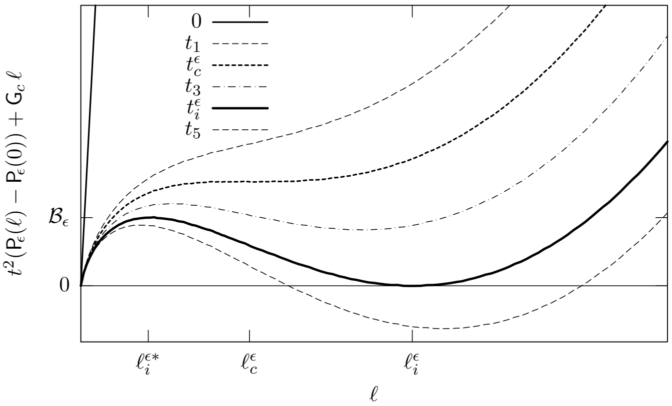

Fig. 7 Graph of $\ell \mapsto (\mathcal{E}_\epsilon(t,\ell) - \mathcal{E}_\epsilon(t,0))$ when $\epsilon = 0.2$ for different values of t. The thick black curve corresponds to the initiation loading $\mathsf{t}_\mathsf{i}^\epsilon$. It gives both the initiation length ℓ_i^ϵ and the energy barrier $\mathcal{B}_\epsilon$ at $\ell_\mathsf{i}^{\epsilon *}$

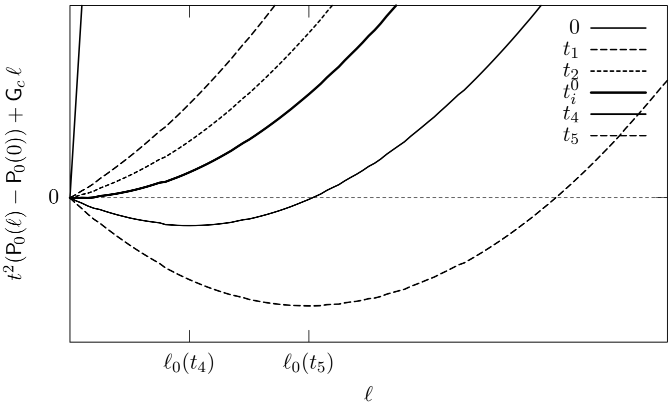

Fig. 8 Graph of $\ell \mapsto (\mathcal{E}_0(t,\ell) - \mathcal{E}_0(t,0))$ for different values of t. The thick black curve corresponds to the initiation loading t_i^0

5 Perspectives

The example of the notch-shaped body confirms the general analysis of Section 2: only the FM-law is really able to account for the initiation of a crack in a sound body, at least in the setting of Griffith's energy assumptions. Moreover, the fact that this law ensures the continuity of the response with respect to the angle of the notch is sufficient to reaffirm its interest. On the other hand, the lack of continuity of Griffith's criterion proves definitively its incapacities. It is certainly possible to extend the main results of this paper to more general situations, like 3D domains and non predefined crack paths. However, the most interesting issue is probably to leave the context of global minimization and to obtain similar results in the more convenient context of local minimization. It is not an easy task, because the elastic response is in general always a local minimum. Thus, the key point is to propose a criterion which allows to jump from a local minimum to another one, under the condition that these local minima are sufficiently close in a certain sense (in terms of energy barrier for example). This type of criterion could be obtained by starting from regularized models like cohesive forces models and by passing to the limit in the regularizing parameter, in the spirit of [31] or [21].

A Differentiability of the potential energy

The proofs of the differentiability of $\ell \mapsto \mathsf{P}_\epsilon(\ell)$ and of the smoothness of $\mathsf{P}_\epsilon(\ell)$ with respect to ϵ are based on a change of variables which sends the parameter dependent domain onto a fix domain. Then, a part of the regularity results are a direct consequence of the following Lemma

Lemma 18 *Let $\mathcal{H}$ be an Hilbert space with norm $\|\cdot\|$ and let Λ be a real interval. Let $\{\mathsf{p}^i\}_{1\le i\le m}$ be a family of continuous bilinear symmetric forms on $\mathcal{H}$ and $\{\mathsf{q}^i\}_{1\le i\le n}$ a family of continuous linear forms on $\mathcal{H}$. Let $\{a^i_\lambda\}_{1\le i\le m}$, $\{b^i_\lambda\}_{1\le i\le n}$ and c_λ be real-valued functions of λ, differentiable in Λ.*

If $\mathsf{p}_\lambda := \sum_{i=1}^m a^i_\lambda \mathsf{p}^i$ is coercive on $\mathcal{H}$, uniformly with respect to λ, i.e.

$$\exists \alpha > 0 \quad \forall \lambda \in \Lambda \qquad \mathsf{p}_\lambda(u,u) \ge \alpha \|u\|^2, \quad \forall u \in \mathcal{H}, \tag{58}$$

then the three following properties hold

1. *For every $\lambda \in \Lambda$, the minimization problem $\min_{u\in\mathcal{H}} \left\{\frac{1}{2}\mathsf{p}_\lambda(u,u) + \mathsf{q}_\lambda(u) + c_\lambda\right\}$, with $\mathsf{q}_\lambda := \sum_{i=1}^n b^i_\lambda \mathsf{q}^i$, admits a unique solution u_λ;*
2. *The minimizer u_λ is a differentiable function of λ on Λ and its derivative $\dot{u}_\lambda \in \mathcal{H}$ is given by*
$$\mathsf{p}_\lambda(\dot{u}_\lambda, v) + \sum_{i=1}^m \dot{a}^i_\lambda \mathsf{p}^i(u_\lambda, v) + \sum_{i=1}^n \dot{b}^i_\lambda \mathsf{q}^i(v) = 0, \quad \forall v \in \mathcal{H}, \tag{59}$$
where the dot denotes the derivative with respect to λ.
3. *The minimum $\mathsf{P}_\lambda := \frac{1}{2}\mathsf{p}_\lambda(u_\lambda, u_\lambda) + \mathsf{q}_\lambda(u_\lambda) + c_\lambda$ is a differentiable function of λ on Λ and its derivative is given by*
$$\dot{\mathsf{P}}_\lambda = \frac{1}{2}\sum_{i=1}^m \dot{a}^i_\lambda \mathsf{p}^i(u_\lambda, u_\lambda) + \sum_{i=1}^n \dot{b}^i_\lambda \mathsf{q}^i(u_\lambda) + \dot{c}_\lambda. \tag{60}$$

Proof. The proof presents no difficulty and we only give the main steps.

1. Existence and uniqueness of u_λ. It is a direct consequence of the coercivity of p_λ and the continuity of q_λ, see [16]. Moreover u_λ satisfies the variational equality

$$\mathsf{p}_\lambda(u_\lambda, v) + \mathsf{q}_\lambda(v) = 0, \quad \forall v \in \mathcal{H}. \tag{61}$$

2. Differentiability of u_λ and characterization of $\dot{u}_\lambda$. Let $v_h := (u_{\lambda+h} - u_\lambda)/h$ with $h \neq 0$ small enough (and with h having the right sign if λ is a bound of Λ). Using the variational equalities (61) satisfied by u_λ and $u_{\lambda+h}$, we get

$$\mathsf{p}_{\lambda+h}(v_h, v) + \sum_{i=1}^m \frac{a^i_{\lambda+h} - a^i_\lambda}{h}\mathsf{p}^i(u_\lambda, v) + \sum_{i=1}^n \frac{b^i_{\lambda+h} - b^i_\lambda}{h}\mathsf{q}^i(v) = 0, \quad \forall v \in \mathcal{H} \tag{62}$$

from which we deduce that the sequence v_h is bounded in $\mathcal{H}$. Hence a subsequence weakly converges in $\mathcal{H}$. Passing to the limit in (62), we obtain that the limit $\dot{u}_\lambda$ satisfies (59) and hence is unique. Therefore, all the sequence v_h weakly converges to $\dot{u}_\lambda$. To prove that it converges strongly, it suffices to prove that $\lim_{h\to 0} \mathsf{p}_\lambda(v_h, v_h) = \mathsf{p}_\lambda(\dot{u}_\lambda, \dot{u}_\lambda)$. Setting $v = v_h$ in (62) and passing to the limit when $h \to 0$, we get

$$\lim_{h\to 0} \mathsf{p}_\lambda(v_h, v_h) = -\sum_{i=1}^m \dot{a}^i_\lambda \mathsf{p}^i(u_\lambda, \dot{u}_\lambda) - \sum_{i=1}^n \dot{b}^i_\lambda \mathsf{q}^i(\dot{u}_\lambda).$$

Setting $v = \dot{u}_\lambda$ in (59), the result follows.

3. Calculation of $\dot{\mathsf{P}}_\lambda$. Differentiating P_λ leads to

$$\dot{\mathsf{P}}_\lambda = \mathsf{p}_\lambda(u_\lambda, \dot{u}_\lambda) + \mathsf{q}_\lambda(\dot{u}_\lambda) + \frac{1}{2}\sum_{i=1}^{m} \dot{a}_\lambda^i \mathsf{p}^i(u_\lambda, u_\lambda) + \sum_{i=1}^{n} \dot{b}_\lambda^i \mathsf{q}^i(u_\lambda) + \dot{c}_\lambda.$$

Using (61) with $v = \dot{u}_\lambda$, we obtain (60). Note that the calculation of $\dot{\mathsf{P}}_\lambda$ does not require the calculation of $\dot{u}_\lambda$ but only that of u_λ. □

Remark 6 *By induction, we can adapt this Lemma to prove that* u_λ *and* P_λ *are differentiable as many times as are the* a_λ^i*'s, the* b_λ^i*'s and* c_λ*. Thus, if these latter functions are indefinitely differentiable, then the former ones too.*

References

1. Barenblatt, G. I.: 1962, 'The mathematical theory of equilibrium cracks in brittle fracture'. *Adv. Appl. Mech.* pp. 55–129.
2. Bilteryst, F. and J.-J. Marigo: 1999, 'Initiation of fibre debonding in the pull-out problem'. *C. R. Acad. Sci. II b* **327**(10), 977–983.
3. Bilteryst, F. and J.-J. Marigo: 2003, 'An energy based analysis of the pull-out problem'. *Eur. J. Mech. A-Solid* **22**(1), 55–69.
4. Bourdin, B., G. A. Francfort, and J.-J. Marigo: 2008, 'The variational approach to fracture'. *Journal of Elasticity* **91**(1-3), 5–148.
5. Bui, H. D.: 1978, *Mécanique de la rupture fragile.* Masson.
6. Chambolle, A., G. A. Francfort, and J.-J. Marigo: 2009, 'Revisiting energy release rates in brittle fracture'. *to appear.*
7. Chambolle, A., A. Giacomini, and M. Ponsiglione: 2008, 'Crack initiation in brittle materials'. *Arch. Ration. Mech. An.* **188**(2), 309–349.
8. Charlotte, M., J. Laverne, and J.-J. Marigo: 2006, 'Initiation of cracks with cohesive force models: a variational approach'. *Eur. J. Mech. A-Solid* **25**(4), 649–669.
9. Cherepanov, G. P.: 1979, *Mechanics of Brittle Fracture.* McGraw-Hill International Book Company.
10. Dal Maso, G., G. A. Francfort, and R. Toader: 2005, 'Quasistatic crack growth in nonlinear elasticity'. *Arch. Ration. Mech. An.* **176**(2), 165–225.
11. Dal Maso, G. and R. Toader: 2002, 'A Model for the Quasi-Static Growth of Brittle Fractures: Existence and Approximation Results'. *Arch. Ration. Mech. An.* **162**, 101–135.
12. Del Piero, G.: 1999, 'One-Dimensional ductile-brittle transition, yielding and structured deformations'. *in: P. Argoul, M. Frémond (Eds.), Proceedings of IUTAM Symposium "Variations de domaines et frontières libres en mécanique", Paris, 1997, Kluwer Academic.*
13. Del Piero, G. and L. Truskinovsky: 2001, 'Macro- and micro-cracking in one-dimensional elasticity'. *Int. J. Solids Struct.* **38**(6), 1135–1138.
14. Destuynder, P. and M. Djaoua: 1981, 'Sur une interprétation mathématique de l'intégrale de Rice en théorie de la rupture fragile'. *Math. Met. Appl. Sc* **3**, 70–87.
15. Dugdale, D. S.: 1960, 'Yielding of steel sheets containing slits'. *J. Mech. Phys. Solids* **8**, 100–108.
16. Ekeland, I. and R. Temam: 1976, *Convex Analysis and Variational Problems.* Amsterdam: North-Holland.
17. Ferdjani, H., R. Abdelmoula, and J.-J. Marigo: 2007, 'Insensitivity to small defects of the rupture of materials governed by the Dugdale model'. *Continuum Mech. Therm.* **19**(3-4), 191–210.
18. Francfort, G. A. and C. Larsen: 2003, 'Existence and convergence for quasi-static evolution in brittle fracture'. *Commun. Pur. Appl. Math.* **56**(10), 1465–1500.
19. Francfort, G. A. and J.-J. Marigo: 1998, 'Revisiting brittle fracture as an energy minimization problem'. *J. Mech. Phys. Solids* **46**(8), 1319–1342.

20. Francfort, G. A. and A. Mielke: 2006, 'Existence results for a class of rate-independent material models with nonconvex elastic energies'. *J. Reine Angew. Math.* **595**, 55–91.
21. Giacomini, A.: 2005, 'Size effects on quasi-static growth of cracks'. *SIAM J. Math. Anal.* **36**(6), 1887–1928 (electronic).
22. Griffith, A.: 1920, 'The phenomena of rupture and flow in solids'. *Phil. Trans. Roy. Soc. London* **CCXXI**(A), 163–198.
23. Grisvard, P.: 1985, *Elliptic problems in non smooth domains*, No. 24 in Monographs and Studies in Mathematics. Pitman.
24. Grisvard, P.: 1986, 'Problèmes aux limites dans les polygones; Mode d'emploi'. *EDF, Bulletin de la Direction des Études et Recherches* **Série C**(1), 21–59.
25. Irwin, G. R.: 1958, 'Fracture'. *In Handlbuchdes Physik, Springer Verlag* **6**, 551–590.
26. Laverne, J. and J.-J. Marigo: 2004, 'Global approach, relative minima and yield criterion in Fracture Mechanics'. *Comptes Rendus Mécanique* **332**(4), 313–318.
27. Lawn, B.: 1993, *Fracture of Brittle Solids - Second Edition*, Cambridge Solid State Science Series. Cambridge: Cambridge University press.
28. Leblond, J.-B.: 2000, *Mécanique de la rupture fragile et ductile*, Collection Études en mécanique des matériaux et des structures. Editions Lavoisier.
29. Leguillon, D.: 1990, 'Calcul du taux de restitution de l'énergie au voisinage d'une singularité'. *C. R. Acad. Sci. II b* **309**, 945–950.
30. Marigo, J.-J.: 1989, 'Constitutive relations in Plasticity, Damage and Fracture Mechanics based on a Work Property'. *Nuclear Engineering and Design* **114**(3), 249–272.
31. Marigo, J.-J. and L. Truskinovky: 2004, 'Initiation and propagation of fracture in the models of Griffith and Barenblatt'. *Continuum Mech. Therm.* **16**(4), 391–409.
32. Marsden, J. E. and T. J. R. Hughes: 1983, *Mathematical fundations of elasticity*. Prentice-Hall.
33. Mielke, A.: 2005, 'Evolution of rate-independent systems'. In: *Evolutionary equations*, Vol. II of *Handb. Differ. Equ.* Amsterdam: Elsevier/North-Holland, pp. 461–559.
34. Muskhelishvili, N. I.: 1963, *Some basic problems of mathematical theory of elasticity*. Groningen: P. Noordhoff Ltd, $2^n d$ edition.
35. Needleman, A.: 1992, 'Micromechanical modelling of interface decohesion'. *Ultramicroscopy* **40**, 203–214.
36. Negri, M.: 2008, 'A comparative analysis on variational models for quasi-static brittle crack propagation'. *preprint*.
37. Negri, M. and C. Ortner: 2008, 'Quasi-static crack propagation by Griffith's criterion'. *Math. Models Methods Appl. Sci.* **18**, 1895–1925.
38. Nguyen, Q. S.: 2000, *Stability and Nonlinear Solid Mechanics*. London: Wiley & Son.
39. Rice, J. R.: 1968, 'A path independent integral and the approximate analysis of strain concentration by notches and cracks'. **35**, 379–386.