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Homogenization and kinetic models in extended phase-space

Abstract. This paper reviews recent results obtained in collaboration with E. Bernard and E. Caglioti on the homogenization problem for the linear Boltzmann equation for a monokinetic population of particles set in a periodically perforated domain, assuming that particles are absorbed by the holes. We distinguish a critical scale for the hole radius in terms of the distance between neighboring holes, derive the homogenized equation under this scaling assumption, and study the asymptotic mass loss rate in the long time limit. The homogenized equation so obtained is set on an extended phase space as it involves an extra time variable, which is the time since the last jump in the stochastic process driving the linear Boltzmann equation. The present paper proposes a new proof of exponential decay for the mass which is based on a priori estimates on the homogenized equation instead of the renewal theorem used in [Bernard-Caglioti-Golse, SIAM J. Math. Anal. **42** (2010), 2082–2113].

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1 - Introduction

Homogenization of a transport process in a periodic system of scatterers (or holes) may lead to nontrivial mathematical phenomena, as some memory of the correlations between obstacles can remain after passing to the limit.

The present paper will discuss the simplest problem leading to such phenomena, i.e. the evolution of a gas of monokinetic point particles governed by a linear Boltzmann equation in a periodic distribution of holes. Particles falling into the holes are removed from the particle system, which therefore loses mass, and this suggests the following question:

To find the appropriate scaling for which the mass loss per unit of time is of order unity, and to determine the mass loss rate in that case.

As we shall see below, the solution of this seemingly simple homogenization problem is surprisingly intricate, at variance with the analogous periodic homogenization problem for a diffusion process, which leads to a conceptually simpler result.

2 - The diffusion case

As a warm-up, we briefly recall the answer to this question when the linear Boltzmann equation is replaced with a diffusion equation.

Let $\epsilon\mathbf{Z}^N$ designate the cubic lattice with mesh size ϵ, and consider the Euclidian space with balls of radius $r_\epsilon \in (0, \epsilon/2)$ centered at lattice points removed:

$$\Omega[\epsilon, r_\epsilon] = \{x \in \mathbf{R}^N \mid \text{ dist } (x, \epsilon\mathbf{Z}^N) > r_\epsilon\}\,.$$

Consider next the Dirichlet problem with unknown $u_\epsilon \equiv u_\epsilon(x) \in \mathbf{R}$:

$$\left\{\begin{array}{l} \lambda u_\epsilon - \Delta u_\epsilon = f\,, \quad x \in \Omega[\epsilon, r_\epsilon]\,, \\ u_\epsilon\big|_{\partial\Omega[\epsilon, r_\epsilon]} = 0\,, \end{array}\right.$$

where $f \in L^2(\mathbf{R}^N)$ and $\lambda > 0$ are given.

For this problem, there is a natural notion of *critical size* of the obstacles:

$$r_\epsilon = r\epsilon^{\gamma_c}\,, \qquad \text{with } \gamma_c = \frac{N}{N-2} \text{ whenever } N > 2\,.$$

The exponent γ_c is critical in the following sense:

if $r_\epsilon \ll \epsilon^{\gamma_c}$, one expects that $u_\epsilon \to (\lambda - \Delta)^{-1} f$,
if $r_\epsilon \gg \epsilon^{\gamma_c}$, one expects that $u_\epsilon \to 0$.

In the critical case, L'vov-Khruslov [17] and Cioranescu-Murat [10] proved that

T h e o r e m 2.1. *If $r_\epsilon \sim r\epsilon^{\gamma_c}$ as $\epsilon \to 0^+$ with $\gamma_c = \frac{N}{N-2}$ for $N > 2$, for each $f \in L^2(\mathbf{R}^N)$, the solution u_ϵ of the Dirichlet problem*

$$\left\{\begin{array}{l} \lambda u_\epsilon - \Delta u_\epsilon = f\,, \quad x \in \Omega[\epsilon, r_\epsilon]\,, \\ u_\epsilon\big|_{\partial\Omega[\epsilon, r_\epsilon]} = 0\,, \end{array}\right.$$

extended by 0 *in the holes, converges in $L^2(\mathbf{R}^N)$ to the solution u of*

$$-\Delta u + (\lambda + c)u = f\,, \quad x \in \mathbf{R}^N\,,$$

where $c > 0$ is a positive number (limiting absorption rate) independent of the data f.

In other words, the effect of the periodic system of holes at the critical size on the limiting equation is given by the enhanced damping coefficient $\lambda + c$ — instead of λ as in the original problem. Cioranescu and Murat coined the phrase *"un terme étrange venu d'ailleurs"* (a strange term coming from elsewhere) to designate this additional damping term.

The 2-dimensional case is somewhat special, as there is no critical *exponent* in that case. Instead, the *critical size* of the holes is $r_\epsilon = Ce^{-1/\epsilon^2}$. With this modification, the result above holds verbatim.

R e m a r k 2.1. *The two following remarks are in order:*
1) In the theorem above, the limiting problem has the same structure as the original problem.
2) The coefficient c is found to be $c = r\,Cap(\mathbf{S}^{N-1})$, where $Cap(\mathbf{S}^{N-1})$ designates the 2-capacity of the unit sphere, defined by the usual variational problem:

$$Cap(\mathbf{S}^{N-1}) = \inf_{\substack{\phi\in H^1(\mathbf{R}^3)\\ |x|\le 1 \Rightarrow \phi(x)\ge 1}} \int_{|x|\ge 1} |\nabla\phi(x)|^2 dx\,.$$

In the context of porous media, the above result can be extended to the Stokes or Navier-Stokes equations (see Allaire [2]): the super-critical scaling $r_\epsilon \gg \epsilon^{\gamma_c}$ leads to the d'Arcy law, whereas the critical case $r_\epsilon \sim r\epsilon^{\gamma_c}$ leads to Brinkman's friction force in the Stokes equation.

Allaire's result was further extended to the case of randomly distributed holes at critical size by Desvillettes, Ricci and the author [12]: the result is similar to the purely periodic case, assuming that the obstacles are well separated — we refer the interested reader to [12] for a precise description of this assumption — except that it involves the empirical distribution of holes.

3 - The distribution of free path lengths

In the sequel, we shall consider a linear Boltzmann equation in space dimension $N \ge 2$ set on the Euclidian space with a periodic system of balls removed:

$$\Omega_\epsilon = \{x \in \mathbf{R}^N \mid \mathrm{dist}(x, \epsilon\mathbf{Z}^2) > \epsilon^{N/(N-1)}\}\,.$$

The holes in the domain Ω_ϵ are at the critical size — with respect to the period ϵ, i.e. the smallest distance between neighboring ball centers. See Figure 3 for the domain Ω_ϵ in space dimension 2.

Observe that, for the transport (or linear Boltzmann) equation — which a 1st order PDE instead of a 2nd order PDE like the Laplace equation — the critical exponent is $\gamma_c = \frac{N}{N-1}$ instead of $\frac{N}{N-2}$ as recalled in the previous section.

Henceforth, we only consider the 2-dimensional problem. Most of our results carry over to all space dimensions, except for the explicit computation of the

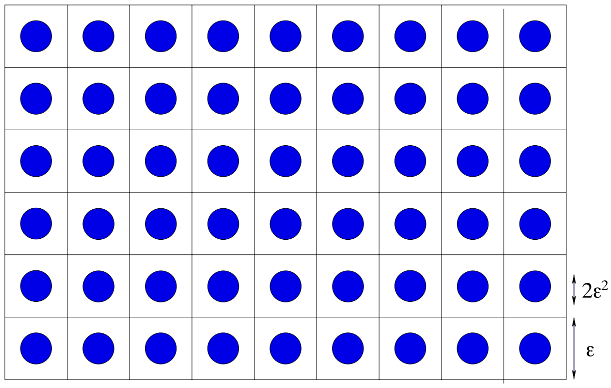

Figure 1: The perforated domain Ω_ϵ with holes at critical size in dimension 2

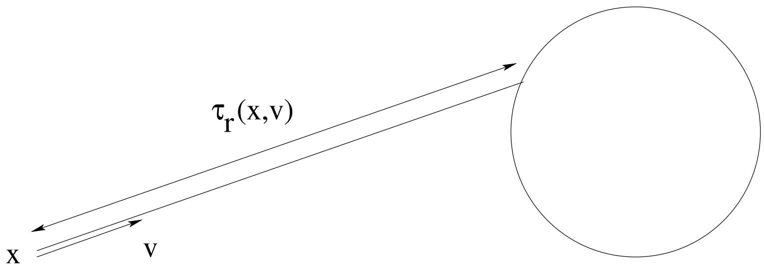

Figure 2: The free path length

distribution of free path lengths which is not known for $N > 2$ (although recent progress by Marklof and Strömbergsson [19] provide an asymptotic equivalent for the tail of that distribution in all space dimensions.)

Before going further, we recall the definition of free path length.

D e f i n i t i o n 3.1. *The free path length for a particle moving at speed* 1 *in the domain* Ω_ϵ*, starting from* $x \in \Omega_\epsilon$ *in the direction* $v \in \mathbf{S}^1$ *is*

$$\tau_\epsilon(x, v) = \min\{t > 0 \,|\, x + tv \in \partial\Omega_\epsilon\} \,.$$

See Figure 3 for the geometric meaning of the free path length.

Dahlqvist conjectured in [11], and Boca and Zaharescu proved in [6] an explicit formula for the distribution of free path lengths

$$\Phi(t) := \lim_{\epsilon \to 0^+} \mathrm{Prob}(\tau_\epsilon > t) \,,$$

assuming that x and ω are independent and uniformly distributed in $\Omega_\epsilon / \epsilon\mathbf{Z}^2$ (equivalently, in any period of Ω_ϵ) and in $\mathbf{S}^1$ respectively.

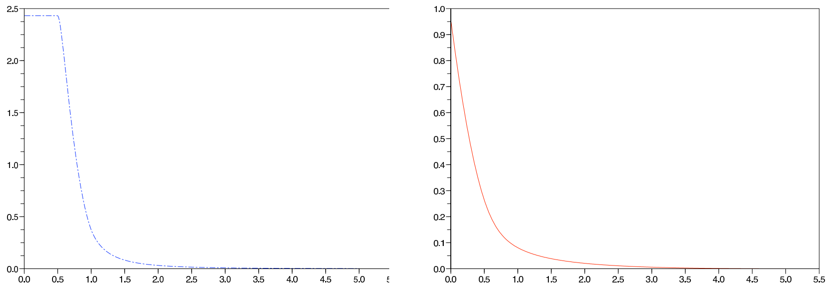

Figure 3: The graphs of $\frac{24}{\pi^2}g$ (left) and Φ (right)

T h e o r e m 3.1 (Boca-Zaharescu, 2007). *For each* $t > 0$

$$\Phi(t) = \tfrac{24}{\pi^2}\int_t^\infty (s-t)g(s)ds\,,$$

where

$$g(s) = \begin{cases} 1 & s \in [0, \frac{1}{2}]\,, \\ \frac{1}{2s} + 2\left(1 - \frac{1}{2s}\right)^2 \ln(1 - \frac{1}{2s}) - \frac{1}{2}\left|1 - \frac{1}{s}\right|^2 \ln|1 - \frac{1}{s}| & s \in (\frac{1}{2}, \infty)\,. \end{cases}$$

The proof of this result is based on Farey fractions, and on a particular 3-term partition of the 2-torus found in [5] and [9] that is reminiscent of the 3-length theorem conjectured by Steinhaus and eventually established by Sós [21] and Suranyi [22]

4 - Homogenization of the linear Boltzmann equation

Consider the linear Boltzmann equation in Ω_ϵ with unknown $f_\epsilon(t, x, v)$, with absorbing boundary condition on the edge of the holes

$$\begin{cases} \partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon + \sigma(f_\epsilon - K f_\epsilon) = 0\,, & x \in \Omega_\epsilon\,,\ |v| = 1 \\ f_\epsilon(t, x, v) = 0\,, & x \in \partial\Omega_\epsilon\,,\ v \cdot n_x > 0 \\ f_\epsilon\big|_{t=0} = f^{in}\,, & \end{cases}$$

where $\sigma > 0$ and K is an integral operator acting on the v-variable only, i.e.

$$Kf(t, x, v) = \int_{\mathbf{S}^1} k(v, w)f(t, x, w)dw\,.$$

The integral kernel of the operator K is a function $k \in C(\mathbf{S}^1 \times \mathbf{S}^1)$ satisfying the assumptions

$$k(v, w) = k(w, v) \geq 0 \quad \text{for each } v, w \in \mathbf{S}^1\,,$$

and

$$\int_{\mathbf{S}^1} k(v,w)dw = 1 \quad \text{ for each } v \in \mathbf{S}^1 \,.$$

Our purpose is to investigate the following problems:

1) to find an equation for the limit of f_ϵ as $\epsilon \to 0^+$, and

2) to find the asymptotic mass loss rate, and especially to check whether or not it is exponential.

We begin with the answer to the first question, which is an analogue for the linear Boltzmann equation of the problem studied by L'vov-Khruslov [17] and Cioranescu-Murat [10], recalled in section 2. The result below was obtained by the author in collaboration with E. Bernard and E. Caglioti [4].

T h e o r e m 4.1 (Bernard-Caglioti-Golse, 2010). *In the limit as $\epsilon \to 0^+$, denoting by f_ϵ both the solution of the linear Boltzmann equation defined a.e. on $\mathbf{R}_+ \times \Omega_\epsilon \times \mathbf{S}^1$ together with its extension by 0 in the holes (i.e. in Ω_ϵ^c), one has*

$$f_\epsilon \to \int_0^\infty F(t,s,x,v)ds \text{ in } L^\infty(\mathbf{R}_+ \times \mathbf{R}^2 \times \mathbf{S}^1) \text{ weak-*},$$

where $F \equiv F(t,s,x,v)$ satisfies, for all $(x,v) \in \mathbf{R}^2 \times \mathbf{S}^1$, $t,s > 0$

$$\begin{cases} \partial_t F + v\cdot\nabla_x F + \partial_s F = -\sigma F + \frac{\Phi'}{\Phi}(s\wedge t)F \,, \\ \qquad\qquad F(t,0,x,v) = \sigma \displaystyle\int_0^\infty KF(t,s,x,v)ds \,, \\ \qquad\qquad F(0,s,x,v) = \sigma e^{-\sigma s} f^{in}(x,v) \,. \end{cases}$$

Unlike in the periodic homogenization problem for the Laplace equation, the homogenization of the linear Boltzmann equation in a periodic system of holes involves an additional variable s (together with the usual variables t,x,v of the kinetic theory of gases). In other words, the homogenized equation is set on a phase-space larger than the phase-space usual in kinetic models. Specifically,

$$\text{phase-space} = \mathbf{R}^2_x \times \mathbf{S}^1_v \times [0,\infty)_s \text{ instead of } \mathbf{R}^2_x \times \mathbf{S}^1_v \,,$$

as in the usual kinetic theory for particles moving at unit speed.

The extra-variable s has the following physical interpretation. As is well known, the probabilistic interpretation of the linear Boltzmann equation involves a jump process in velocity, where the jump times are exponentially distributed. The variable s is the time since the last jump in velocity — in other words, $F(t,s,x,v)$ is the conditional expectation of the particle density knowing the time of the last jump in velocity.

The theorem above can be summarized by the following diagram

$$\begin{array}{ccccc} & F_\epsilon & \xrightarrow{\text{Homogenization}} & F & \\ \text{lifting of initial data} & \Big\uparrow & & \Big\downarrow & \text{integration in } s \\ & f_\epsilon & & f & \end{array}$$

Perhaps the most remarkable feature in the theorem above is that the homogenization process, which is a *macroscopic* limit of the linear Boltzmann equation, triggers the dependence of the particle distribution function in the extra *microscopic* variable s, although this variable is not present in the original equation. In other words, it is quite surprising that a macroscopic limit of the linear Boltzmann equation should involve a more microscopic level of description than the original problem.

5 - Asymptotic mass loss rate

Now we turn to the second question raised above, i.e. finding the asymptotic mass loss rate in the long time limit.

A first step is to condition the mass at time t by the s variable; in other words, we consider $m(t,s)$ to be the density at time t of particles whose velocities (in the stochastic process driving the linear Boltzmann equation) have jumped at time $t-s$.

P r o p o s i t i o n 5.1 (Bernard-Caglioti-Golse [4]). *One has*

$$\tfrac{1}{2\pi}\iint_{\mathbf{R}^2\times\mathbf{S}^1} f_\epsilon(t,x,v)dxdv \to M(t)=\int_0^\infty m(t,s)ds \ \textit{a.e. in } t\geq 0\,,$$

as $\epsilon\to 0^+$, *where* $m\equiv m(t,s)$ *is the solution of*

$$\begin{cases} \partial_t m(t,s)+\partial_s m(t,s)+B(t,s)m=0\,, & t,s>0\,,\\ m(t,0)=\sigma\displaystyle\int_0^\infty m(t,s)ds\,, & t>0\,,\\ m(0,s)=\sigma e^{-\sigma s}M(0)\,, & s>0\,,\end{cases}$$

and where

$$B(t,s){:=}\sigma-\tfrac{\Phi'}{\Phi}(s\wedge t)\,,\qquad \textit{while}\ \ M(0)=\tfrac{1}{2\pi}\iint_{\mathbf{R}^2\times\mathbf{S}^1} f^{in}(x,v)dxdv\,.$$

A somewhat surprising consequence of the structure of the equation governing m is that, in the long time limit, the total mass (=total number of particles) decays exponentially fast.

T h e o r e m 5.1 (Bernard-Caglioti-Golse [4]). *For each* $\sigma>0$, *there exists a unique* $\xi_\sigma\in(-\sigma,0)$ *such that*

$$M(t)\sim C_\sigma e^{\xi_\sigma t} \ \textit{as } t\to+\infty$$

where

$$C_\sigma = \frac{\frac{1}{2\pi}\iint_{\mathbf{R}^2\times\mathbf{S}^1} f^{in}(x,v)dxdv}{\sigma\int_0^\infty s\Phi(s)e^{-(\sigma+\xi_\sigma)s}ds}\,.$$

Besides

$$\xi_\sigma \sim -\sigma \textit{ as } \sigma\to 0^+\,,\ \textit{and}\quad \xi_\sigma\to -2 \textit{ as } \sigma\to+\infty\,.$$

Several remarks are in order.

First this asymptotic behavior is non uniform as $\sigma\to 0$. In the case $\sigma=0$ (i.e. in the case of the free transport equation in a periodic system of absorbing holes), the mass loss rate is given by the asymptotic behavior of the function Φ:

$$M(t) = \Phi(t)\tfrac{1}{2\pi}\iint_{\mathbf{R}^2\times\mathbf{S}^1} f^{in}(x,v)dxdv\,,$$

so that

$$M(t)\sim\frac{M(0)}{\pi^2 t}\quad \text{ as } t\to+\infty\,.$$

Secondly, the asymptotic equivalent of the distribution of free path lengths $\Phi(s)$ as $s\to+\infty$, i.e.

$$\Phi(s)\sim\frac{1}{\pi^2 s}\quad\text{ as } s\to+\infty$$

was known before the explicit computation by Boca-Zaharescu [6] recalled in Theorem 3.1 above: see [9]. In fact, it was proved in [8] and [16] that, in any space dimension, the distribution of free path lengths satisfies bounds of the form

$$\frac{C'}{s}\le \operatorname{Prob}(\tau_\epsilon > s)\le\frac{C}{s}\,,\quad s>1$$

for positive constants $C>C'$ uniformly as $\epsilon\to 0$. A recent result by Marklof-Strömbergsson [19] establishes that

$$\Phi(s) := \lim_{\epsilon\to 0}\operatorname{Prob}(\tau_\epsilon > s)\sim\frac{\pi^{\frac{N-1}{2}}}{2^N N\Gamma(\frac{N+3}{2})\zeta(N)}\frac{1}{s}$$

as $s\to+\infty$.

Finally, it should be mentioned that E. Bernard [3] proved exponential decay of the total mass

$$\tfrac{1}{2\pi}\iint_{\Omega_\epsilon\times\mathbf{S}^1} f_\epsilon(t,x,v)dxdv$$

uniformly in $\epsilon>0$, before the structure of the limiting, homogenized equation was fully understood.

The collisionless case $\sigma=0$ was considered by E. Caglioti and the author in [9]. The main result obtained there is as follows:

T h e o r e m 5.2 (Caglioti-Golse, 2003). *Let f_ϵ satisfy*

$$\left\{\begin{array}{ll} \partial_t f_\epsilon + v\cdot\nabla_x f_\epsilon = 0\,, & x\in\Omega_\epsilon\,,\ |v|=1\,,\\ f_\epsilon(t,x,v)=0\,, & x\in\partial\Omega_\epsilon\,,\ v\cdot n_x>0\,,\\ f_\epsilon\big|_{t=0}=f^{in}\,. & \end{array}\right.$$

Then, for each $t>0$, f_ϵ converges in the sense of Cesàro

$$\frac{1}{|\ln\epsilon|}\int_\epsilon^{1/4} f_r(t,x,v)\frac{dr}{r}\to f(t,x,v)\ \textit{a.e. in } (x,v)\in\mathbf{R}^2\times\mathbf{S}^1$$

in the limit as $\epsilon\to 0^+$, where

$$\left\{\begin{array}{ll} (\partial_t + v\cdot\nabla_x) f(t,x,v) = \frac{\Phi'}{\Phi}(t) f(t,x,v)\,, & x\in\mathbf{R}^2\,,\ |v|=1\,,\\ f\big|_{t=0}=f^{in}\,. & \end{array}\right.$$

In the limit as $t\to+\infty$ one has $\frac{\Phi'(t)}{\Phi(t)}\sim-\frac{1}{t}$.

(In fact, the Cesàro average in r is unnecessary: see Lemma 6.1 below, which is itself a consequence of the main result in [6] in space dimension 2, and of [18] in higher space dimensions.)

In other words, in the non collisional case, the limiting density f satisfies itself an equation of the same type as the original free transport equation — except that the limiting equation is non autonomous. Therefore, the need for an extended phase-space, with the additional variable s, arises only for the collisional case $\sigma>0$ — so that the result in Theorem 5.2 fails to provide any intuition about the structure of the homogenized equation in the collisional case $\sigma>0$.

6 - Sketch of the proof of Theorem 4.1

How the extra variable s appear in the limiting equation obtained in Theorem 4.1 may seem somewhat mysterious. The present section gives the main ideas used in the proof of that result.

6.1 - *Step 1: lifting the Boltzmann equation.*

The key idea is to introduce the variable s already at the level of the linear Boltzmann equation (before passing to the limit as $\epsilon\to 0^+$). Thus, the problem

$$\left\{\begin{array}{rl} \partial_t f_\epsilon + v\cdot\nabla_x f_\epsilon + \sigma(f_\epsilon - Kf_\epsilon) = 0\,, & x\in\Omega_\epsilon\,,\ |v|=1\,,\\ f_\epsilon(t,x,v)=0\,, & x\in\partial\Omega_\epsilon\,,\ v\cdot n_x>0\,,\\ f_\epsilon\big|_{t=0}=f^{in}\,, & \end{array}\right.$$

is equivalent to

$$f_\epsilon(t,x,v)=\int_0^\infty F_\epsilon(t,s,x,v)ds\,,$$

where F_ϵ is the solution of the *lifted Boltzmann equation*

$$\begin{cases} \partial_t F_\epsilon + v\cdot\nabla_x F_\epsilon + \partial_s F_\epsilon + \sigma F_\epsilon = 0\,, & x\in\Omega_\epsilon\,,\ |v|=1\,,\ t,s>0\,,\\ F_\epsilon(t,s,x,v)=0\,, & x\in\partial\Omega_\epsilon\,,\ v\cdot n_x>0\,,\ t,s>0\,,\\ F_\epsilon(t,0,x,v)=\sigma\displaystyle\int_0^\infty KF_\epsilon(t,s,x,v)ds\,, & x\in\Omega_\epsilon\,,\ |v|=1\,,\ t>0\,,\\ F_\epsilon(0,s,x,v)=f^{in}(x,v)\Pi(s)\,, & x\in\Omega_\epsilon\,,\ |v|=1\,,\ s>0\,. \end{cases}$$

Here, $\Pi\equiv\Pi(s)$ is *any* probability density on the half-line $\mathbf{R}_+$ (for instance $\Pi(s)=\sigma e^{-\sigma s}$.)

There is no hidden difficulty in this statement, which can be proved by inspection.

R e m a r k 6.1. *Given* Π*, the function* F_ϵ *is unique. Even though the function* F_ϵ *itself depends on the specific choice of* Π*, one easily checks that* f_ϵ *is independent of the choice of* Π.

6.2 - *Step 2: finding an explicit formula for* F_ϵ*.*

Solving for F_ϵ by the method of characteristics leads to the explicit formula

$$F_\epsilon(t,s,x,v)=F_{1,\epsilon}(t,s,x,v)+F_{2,\epsilon}(t,s,x,v)\,,$$

where

$$\begin{cases} F_{1,\epsilon}(t,s,x,v)=\mathbf{1}_{s<t}\sigma e^{-\sigma s}Kf_\epsilon(t-s,x-sv,v)\mathbf{1}_{\epsilon\tau_\epsilon(x/\epsilon,v)>s}\,,\\ F_{2,\epsilon}(t,s,x,v)=\mathbf{1}_{t<s}e^{-\sigma t}\Pi(s)f^{in}(x-tv,v)\mathbf{1}_{\epsilon\tau_\epsilon(x/\epsilon,v)>t}\,. \end{cases}$$

6.3 - *Step 3: passing to the limit in* $\mathbf{1}_{\epsilon\tau_\epsilon(x/\epsilon,v)>t}$*.*

We begin with lemma that is an amplification of the convergence

$$\mathrm{Prob}(\tau_\epsilon>t)\to\Phi(t)\quad \text{ as } \epsilon\to 0^+\,,$$

established in [6].

L e m m a 6.1 (Bernard-Caglioti-Golse, 2010). *In the limit as* $\epsilon\to 0^+$*,*

$$\mathbf{1}_{\tau_\epsilon(x,v)\geq t}\to\Phi(t) \text{ in } L^\infty(\mathbf{R}^2\times\mathbf{S}^1) \text{ weak-* for each } t>0\,.$$

See [4] for a proof of this result.

At this point, one extends F_ϵ and f_ϵ by 0 inside the holes, and calls $\{F_\epsilon\}$ and $\{f_\epsilon\}$ the densities so extended.

Clearly, as a straightforward consequence of Lemma 6.1,

$$\{F_{2,\epsilon}\}(t,s,x,v)\to\mathbf{1}_{t<s}e^{-\sigma t}\Pi(s)f^{in}(x-tv,v)\Phi(t)$$

in $L^\infty(\mathbf{R}_+\times\mathbf{R}^2\times\mathbf{S}^1)$ weak-* as $\epsilon\to 0$.

The main difficulty in the problem is therefore to pass to the limit in the product

$$Kf_\epsilon(t-s,x-sv,v)\mathbf{1}_{\epsilon\tau_\epsilon(x/\epsilon,v)>s}\,.$$

6.4 - *Step 4: compactness by velocity averaging.*

On the other hand

$$(\partial_t + v\cdot\nabla_x)\{f_\epsilon\} = (v\cdot n_x)f_\epsilon\big|_{\partial\Omega_\epsilon\times\mathbf{S}^1}\delta_{\partial\Omega_\epsilon} + O(1)_{L^\infty}\,,$$

so that

$$\left\{\begin{aligned} &\|f_\epsilon\|_{L^\infty}\le C \quad \text{(by the maximum principle)},\\ &(\partial_t + v\cdot\nabla_x)\{f_\epsilon\} = O(1)_{\mathcal{M}([0,T]\times B(0,R)\times\mathbf{S}^1)}\,,\end{aligned}\right.$$

where $\mathcal{M}(X)$ designates the space of bounded Radon measures on X. That the holes are at the critical size in the problem considered here is reflected precisely in the second bound above.

The key is to use these estimates together with compactness by velocity averaging. We recall that, since the transport equation is hyperbolic, L^p bounds on $(\partial_t + v\cdot\nabla_x)\{f_\epsilon\}$ do not entail strong relative compactness on $\{f_\epsilon\}$ itself in L^p_{loc}. However, such bounds do imply strong relative compactness in L^p_{loc} on moments (averages of $\{f_\epsilon\}$ in velocity) of the form

$$\int_{\mathbf{S}^1}\{f_\epsilon\}(t,x,v)\phi(v)dv$$

for each $\phi\in L^\infty(\mathbf{S}^1)$. See [1], [15] and [14] for the first results in that direction.

In the specific situation considered in the present paper, we apply Theorem 1.8 on p. 29 of [7], and finds that, for each $1\le p<\infty$

$\{Kf_\epsilon\} = K\{f_\epsilon\}$ is **strongly** relatively compact in $L^p_{loc}(\mathbf{R}_+\times\mathbf{R}^2\times\mathbf{S}^1)$.

Therefore, by weak+strong convergence

$$F_{1,\epsilon}\to \mathbf{1}_{s<t}\sigma e^{-\sigma s}Kf(t-s,x-sv,v)\Phi(s)$$

in $L^p_{loc}(\mathbf{R}_+\times\mathbf{R}^2\times\mathbf{S}^1)$ for each $p\in[1,\infty)$.

The remaining part of the proof is routine.

7 - Proof of exponential decay of the total mass

The equation for m is obtained by integrating the equation for F in x and v, since

$$m(t,s) := \tfrac{1}{2\pi}\iint_{\mathbf{R}^2\times\mathbf{S}^1}F(t,s,x,v)dxdv\,.$$

At this point, the key observation is that, by applying the method of characteristics to the transport equation satisfied by m, the total mass at time t, i.e.

$$M(t) = \int_0^\infty m(t,s)ds$$

satisfies the integral equation of renewal type

$$M(t) = \sigma e^{-\sigma t}\Phi(t) + \int_0^t \sigma e^{-\sigma(t-s)}\Phi(t-s)M(s)ds\,, \quad t \geq 0\,,$$

with defective kernel $e^{-\sigma s}\Phi(s)$. In other words, one has

$$0 \leq e^{-\sigma s}\Phi(s) \text{ for each } s > 0 \text{ and } \int_0^\infty e^{-\sigma s}\Phi(s)ds < 1\,.$$

Applying the classical results on the asymptotic behavior of integral equations of renewal type (see for instance Theorem 2 on p. 349 in Feller's classical treatise [13]), one finds that

$$M(t) \sim C_\sigma e^{\xi_\sigma t} \quad \text{ as } t \to +\infty$$

where ξ_σ is defined as the only solution of

$$\int_0^\infty e^{-(\sigma+\xi)t}\Phi(t)dt = \frac{1}{\sigma}\,, \quad -\sigma < \xi_\sigma < 0\,,$$

while

$$C_\sigma := \frac{c_\sigma}{2\pi}\iint_{\mathbf{R}^2\times\mathbf{S}^1} f^{in}(x,v)dxdv$$

with

$$\frac{1}{c_\sigma} = \int_0^\infty \sigma e^{-(\sigma+\xi_\sigma)t}t\Phi(t)dt\,.$$

If one is not interested in obtaining the exact asymptotic equivalent, but only in the exponential decay of M, one can avoid using the renewal theorem in [13] and proceed with elementary a priori estimates, as follows.

L e m m a 7.1. *For $C \geq |\dot\Phi(2)|^{-1}$, one has*

$$-\frac{\dot\Phi(\tau)}{\Phi(\tau)} \geq \frac{1}{C+\tau}\,, \quad \textit{for each } \tau \geq 0\,.$$

P r o o f. According to the first formula on p. 427 in [6], one has

$$\Phi(\tau) = \sum_{n\geq 1}\phi_n\tau^{-n}\,, \quad \tau \geq 2\,,$$

with $\phi_n > 0$ for each $n \geq 1$ and $\phi_{n+1}/\phi_n \to 2$ as $n \to +\infty$, so that

$$-\dot\Phi(\tau) = \sum_{n\geq 1} n\phi_n\tau^{-n-1} \geq \frac{\Phi(\tau)}{\tau}\,, \quad \tau > 2\,.$$

On the other hand, Φ is decreasing and convex, so that

$$-\dot\Phi(\tau) \geq -\dot\Phi(2) = -\dot\Phi(2)\Phi(0) \geq -\dot\Phi(2)\Phi(\tau) \quad \text{ whenever } \tau \in [0,2]\,.$$

Putting these two inequalities together leads to

$$(|\dot{\Phi}(2)|^{-1}+\tau)|\dot{\Phi}(\tau)| \geq \Phi(\tau) \quad \text{ for each } \tau \in \mathbf{R}_+ ,$$

which is the desired inequality. $\square$

With the inequality in Lemma 7.1, we proceed as follows. Obviously $m \geq 0$, so that

$$\begin{aligned} 0 &= \partial_t m(t,s) + \partial_s m(t,s) + B(t,s)m(t,s) \\ &\geq \partial_t m(t,s) + \partial_s m(t,s) + \left(\sigma + \frac{1}{C + t \wedge s}\right) m(t,s) \\ &\geq \partial_t m(t,s) + \partial_s m(t,s) + \left(\sigma + \frac{1}{C+s}\right) m(t,s) , \end{aligned}$$

with $C = |\dot{\Phi}(2)|^{-1}$.

Another key computation is the following one: for each $\psi \in C^1(\mathbf{R}_+)$ decaying exponentially fast as $s \to +\infty$, one has

$$\begin{aligned} \frac{d}{dt}\int_0^\infty m(t,s)\psi(s)ds &= \int_0^\infty \psi(s)(-\partial_s m(t,s) - B(t,s)m(t,s))ds \\ &= m(t,0)\psi(0) + \int_0^\infty \psi(s)(-\partial_s m(t,s) - B(t,s)m(t,s))ds \\ &= \int_0^\infty m(t,s)(\psi'(s) - B(t,s)\psi(s) + \sigma\psi(0))ds . \end{aligned}$$

Therefore, choose ψ so that

$$\begin{cases} \psi'(s) - \left(\sigma + \dfrac{1}{C+s}\right)\psi(s) + \sigma\psi(0) = -\alpha\psi(s) , \quad s > 0 , \\ \psi(0) = 1 , \end{cases}$$

where $\alpha \in (0, \sigma)$ is chosen so that

$$J := \int_0^\infty \sigma e^{(\alpha-\sigma)\tau} \frac{Cd\tau}{C+\tau} < 1 .$$

Solving for ψ the differential equation above leads to

$$\begin{aligned} \psi(s) &= e^{(\sigma-\alpha)s}\frac{C+s}{C}\left(1 - \int_0^s \sigma e^{(\alpha-\sigma)\tau}\frac{Cd\tau}{C+\tau}\right) \\ &\geq e^{(\sigma-\alpha)s}\frac{C+s}{C} J \geq J > 0 . \end{aligned}$$

With this choice of ψ, one has

$$\frac{d}{dt}\int_0^\infty m(t,s)\psi(s)ds \leq -\alpha \int_0^\infty m(t,s)\psi(s)ds$$

since

$$B(t,s) \geq \left(\sigma + \frac{1}{C+s}\right),$$

and therefore

$$\begin{aligned} JM(t) \leq \int_0^\infty m(t,s)\psi(s)ds &\leq e^{-\alpha t}\int_0^\infty m(0,s)\psi(s)ds \\ &= e^{-\alpha t}\int_0^\infty \sigma e^{-\alpha t}\frac{C+s}{C}ds\,. \end{aligned}$$

Summarizing the argument above, which is a simplified variant of the discussion of the asymptotic behavior of solutions of the renewal equation in [20], we have proved the following:

P r o p o s i t i o n 7.1. *Set $C = |\dot{\Phi}(2)|^{-1}$ and let $\alpha \in (0,\sigma)$ be such that*

$$\int_0^\infty \sigma e^{(\alpha-\sigma)\tau}\frac{Cd\tau}{C+\tau} < 1\,.$$

Then the total mass of the particle system decays as follows:

$$M(t) = O(e^{-\alpha t}) \quad \text{as } t \to +\infty\,.$$

8 - Some open questions

While the asymptotic behavior of the total mass is the first obvious question to be answered, it would be interesting to investigate the long time behavior of the solution $F \equiv F(t,s,x,v)$ of the homogenized equation.

For instance, assume that the initial data is periodic with period 1 in the space variable x, and that $\epsilon = 1/n$. Then the following question arises naturally:

Does there exists $\alpha \in \mathbf{R}$ and $\Psi \equiv \Psi(s,x,v)$ such that $e^{-\alpha t}F_\epsilon(t,s,x,v) \sim \Psi(s,x,v)$ as $t \to +\infty$?

If $\Psi \equiv \Psi(s,v)$ is sought independent of x, then it should satisfy

$$\begin{cases} -\alpha\Psi + \partial_s\Psi = -\sigma\Psi + \frac{\Phi'}{\Phi}(s)\Psi\,, & v \in \mathbf{S}^1\,,\ s > 0\,, \\ \Psi(0,v) = \sigma\displaystyle\int_0^\infty K\Psi(s,v)ds\,, & v \in \mathbf{S}^1\,. \end{cases}$$

It is found that

$$\Psi(s,v) = \psi(v)e^{-(\sigma-\alpha)s}\Phi(s)\,,$$

with

$$\psi(v) = \sigma K\psi(v)\int_0^\infty e^{-(\sigma-\alpha)s}\Phi(s)ds\,.$$

An obvious solution is

$$\psi = K\psi \quad \text{and } 1 = \sigma\int_0^\infty e^{-(\sigma-\alpha)s}\Phi(s)ds$$

In that case ψ is a constant in v, while the second equation above has $\alpha = \xi_\sigma < 0$ as unique solution.

This suggests the following line of investigation:

a) Is that solution Ψ the only possible one?
b) Does the solution F of the homogenized problem satisfy

$$F(t,s,x,v) \to \sigma e^{-(\sigma-\alpha)s}\Phi(s) \iint_{\mathbf{T}^2\times\mathbf{S}^1} f^{in}(x,v)dxdv$$

in some sense to be made precise as $t \to +\infty$?
c) If so, what is the convergence rate?
d) Is there a Lyapunov functional for the homogenized problem?

In the case of a random, Poisson distribution of holes, all the results established above hold verbatim with

$$\Phi(s) = e^{-\lambda s}\,, \qquad \text{for some } \lambda > 0\,.$$

In that case, the additional coefficient

$$\frac{\dot{\Phi}}{\Phi}(t \wedge s) = -\lambda$$

in the homogenized equation obtained in Theorem 4.1 and in Proposition 5.1 is constant, so that the extra variable s can be averaged out, and the limiting distribution function

$$f(t,x,v) = \int_0^\infty F(t,s,x,v)ds$$

satisfies the linear Boltzmann equation

$$\partial_t f + v\cdot\nabla_x f + \sigma(f - Kf) + \lambda f = 0\,.$$

In other words, in the random, Poisson case, the presence of the holes leads to an additional exponential damping term in the linear Boltzmann equation — exactly as in the diffusion case recalled in section 2. In that case, the questions a)-d) above are answered by the spectral gap property for the linear Boltzmann operator in the torus.

Therefore, the questions a)-d) above corresponds with a generalization of that spectral gap property for the kinetic theory in extended phase space obtained in Theorem 4.1. These questions are left for future investigations. The general issue of analyzing renewal equations with energy estimates based on some Lyapunov functional follows the strategy proposed by Perthame in his monograph [20].

Finally, we feel that the general idea of using extended phase spaces to study other singular limit problems in PDEs should perhaps be given more attention.

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