Volatility made observable at last

Michel Fliess[1], Cédric Join[2,3], Frédéric Hatt[4]

[1]LIX (CNRS, UMR 7161), École polytechnique, 91228 Palaiseau, France
Michel.Fliess@polytechnique.edu

[2]CRAN (CNRS, UMR 7039), Nancy-Université, BP 239, 54506 Vandœuvre-lès-Nancy, France
Cedric.Join@cran.uhp-nancy.fr

[3]Équipe Non-A, INRIA Lille – Nord-Europe, France

[4] Lucid Capital Management, 2 avenue Charles de Gaulle, BP 351, 2013 Luxembourg, Luxembourg
hatt@lucid-cap.com

Abstract— The Cartier-Perrin theorem, which was published in 1995 and is expressed in the language of *nonstandard analysis*, permits, for the first time perhaps, a clear-cut mathematical definition of the *volatility* of a financial asset. It yields as a byproduct a new understanding of the means of returns, of the beta coefficient, and of the Sharpe and Treynor ratios. New estimation techniques from automatic control and signal processing, which were already successfully applied in quantitative finance, lead to several computer experiments with some quite convincing forecasts.

Keywords—Time series, quantitative finance, trends, returns, volatility, beta coefficient, Sharpe ratio, Treynor ratio, forecasts, estimation techniques, numerical differentiation, nonstandard analysis.

I. Introduction

Although *volatility*, which reflects the price fluctuations, is ubiquitous in quantitative finance (see, *e.g.*, [3], [18], [22], [28], [32], [37], and the references therein), Paul Wilmott writes rightly ([37], chap. 49, p. 813):

Quite frankly, we do not know what volatility currently is, never mind what it may be in the future.

Our title is explained by sentences like the following one in Tsay's book ([35], p. 98):

... volatility is not directly observable ...

The lack moreover of any precise mathematical definition leads to multiple ways for computing volatility which are by no means equivalent and might even be sometimes misleading (see, *e.g.*, [20]). Our theoretical formalism and the corresponding computer simulations will confirm what most practitioners already know. It is well expressed by Gunn ([21], p. 49):

Volatility is not only referring to something that fluctuates sharply up and down but is also referring to something that moves sharply in a sustained direction.

The existence of trends [11] for time series, which should be viewed as the *means*, or *averages*, of those series, yields

- a natural and straightforward model-free definition of the variance (resp. covariance) of one (resp. two) time series,
- simple forecasting techniques which are based on similar techniques to those in [11], [12], [13], [14].

Exploiting the above approach to volatility for the return of some financial asset necessitates some care due to the highly fluctuating character of returns. This is accomplished by considering the means of the time series associated to the prices logarithms. The following results are derived as byproducts:

1. We complete [13] with a new definition of the classic beta coefficient for returns. It should bypass most of the existing criticisms.
2. The Sharpe ([30], [31]) and Treynor ratios, which are famous performance measures for trading strategies (see, *e.g.*, [3], [28], [34], [37], and the references therein), are connected to a quite arbitrary financial time series. They might lead to new and useful trading *indicators*.

Remark 1 : The graphical representation of all the above quantities boils down to the drawing of means which has been already successfully achieved in [11], [12], [13], [14].

Our paper is organized as follows. After recalling the Cartier-Perrin theorem [6], Section II defines (co)variances and volatility. In order to apply this setting to financial returns, Section III defines the means of returns and suggests definitions of the beta coefficient, and of the Sharpe and Treynor ratios. Numerous quite convincing computer experiments are shown in Section IV, which displays also excellent forecasts for the volatility. Some short discussions on the concept of volatility may be found in Section V.

II. Mean, variance and covariance revisited

A. Time series via nonstandard analysis

A.1 Infinitesimal sampling

Take the time interval $[0,1] \subset \mathbb{R}$ and introduce as often in *nonstandard analysis* the infinitesimal sampling

$$\mathfrak{T} = \{0 = t_0 < t_1 < \cdots < t_\nu = 1\}$$

where $t_{i+1} - t_i$, $0 \leq i < \nu$, is *infinitesimal*, *i.e.*, "very small".[1] A *time series* $X(t)$ is a function $X : \mathfrak{T} \to \mathbb{R}$.

A.2 S-integrability

The *Lebesgue measure* on $\mathfrak{T}$ is the function m defined on $\mathfrak{T}\backslash\{1\}$ by $m(t_i) = t_{i+1} - t_i$. The measure of any interval $[c,d[\subset \mathfrak{I}$, $c \leq d$, is its length $d - c$. The *integral* over $[c,d[$ of the time series $X(t)$ is the sum

$$\int_{[c,d[} X dm = \sum_{t \in [c,d[} X(t) m(t)$$

[1]See, *e.g.*, [7], [8] for basics in nonstandard analysis.

X is said to be *S-integrable* if, and only if, for any interval $[c, d[$ the integral $\int_{[c,d[} |X| dm$ is *limited*[2] and, if $d - c$ is infinitesimal, also infinitesimal.

A.3 Continuity and Lebesgue integrability

X is *S-continuous* at $t_\iota \in \mathfrak{T}$ if, and only if, $f(t_\iota) \simeq f(\tau)$ when $t_\iota \simeq \tau$.[3] X is said to be *almost continuous* if, and only if, it is S-continuous on $\mathfrak{T} \setminus R$, where R is a *rare* subset.[4] X is *Lebesgue integrable* if, and only if, it is S-integrable and almost continuous.

A.4 Quick fluctuations

A time series $\mathcal{X} : \mathfrak{T} \to \mathbb{R}$ is said to be *quickly fluctuating*, or *oscillating*, if, and only if, it is S-integrable and $\int_A \mathcal{X} dm$ is infinitesimal for any *quadrable* subset.[5]

A.5 The Cartier-Perrin theorem

Let $X : \mathfrak{T} \to \mathbb{R}$ be a S-integrable time series. Then, according to the Cartier-Perrin theorem [6],[6] the additive decomposition

$$\boxed{X(t) = E(X)(t) + X_{\text{fluctuation}}(t)} \tag{1}$$

holds where

- the *mean*, or *average*, $E(X)(t)$ is Lebesgue integrable,[7]
- $X_{\text{fluctuation}}(t)$ is quickly fluctuating.

The decomposition (1) is unique up to an infinitesimal.

Remark 2 : $E(X)(t)$, which is "smoother" than $X(t)$, provides a mathematical justification [11] of the *trends* in *technical analysis* (see, *e.g.*, [2], [25]).

Remark 3 : Calculations of the means and of its derivatives, if they exist, are deduced, via new estimation techniques, from the denoising results in [17], [27] (see also [19]), which extend the familiar moving averages, which are classic in technical analysis (see, *e.g.*, [2], [25]).

B. Variances and covariances

B.1 Squares and products

Take two S-integrable time series $X(t)$, $Y(t)$, such that their squares and the squares of $E(X)(t)$ and $E(Y)(t)$ are also S-integrable. The Cauchy-Schwarz inequality shows that the products

- $X(t)Y(t)$, $E(X)(t)E(Y)(t)$,
- $E(X)(t)Y_{\text{fluctuation}}(t)$, $X_{\text{fluctuation}}(t)E(Y)(t)$,
- $X_{\text{fluctuation}}(t)Y_{\text{fluctuation}}(t)$

are all S-integrable.

B.2 Differentiability

Assume moreover that $E(X)(t)$ and $E(Y)(t)$ are *differentiable* in the following sense: there exist two Lebesgue integrable time series $f, g : \mathfrak{T} \to \mathbb{R}$, such that, $\forall\, t \in \mathfrak{T}$, with the possible exception of a limited number of values of t, $E(X)(t) = E(X)(0) + \int_0^t f(\tau)d\tau$, $E(Y)(t) = E(Y)(0) + \int_0^t g(\tau)d\tau$. Integrating by parts shows that the products $E(X)(t)Y_{\text{fluctuation}}(t)$ and $X_{\text{fluctuation}}(t)E(Y)(t)$ are quickly fluctuating [9].

Remark 4 : Let us emphasize that the product

$$X_{\text{fluctuation}}(t)Y_{\text{fluctuation}}(t)$$

is not necessarily quickly fluctuating.

B.3 Definitions

1. The *covariance* of two time series $X(t)$ and $Y(t)$ is

$$\begin{aligned} \text{cov}(XY)(t) &= E\left((X - E(X))(Y - E(Y))\right)(t) \\ &\simeq E(XY)(t) - E(X)(t) \times E(Y)(t) \end{aligned}$$

2. The *variance* of the time series $X(t)$ is

$$\begin{aligned} \text{var}(X)(t) &= E\left((X - E(X))^2\right)(t) \\ &\simeq E(X^2)(t) - (E(X)(t))^2 \end{aligned}$$

3. The *volatility* of $X(t)$ is the corresponding standard deviation

$$\boxed{\text{vol}(X)(t) = \sqrt{\text{var}(X)(t)}} \tag{2}$$

The volatility of a quite arbitrary time series seems to be precisely defined here for the first time.

Remark 5 : Another possible definition of the volatility (see [20]), which is not equivalent to Equation (2), is the following one

$$E\left(|X - E(X)|\right)(t)$$

It will not be exploited here.

III. Returns

A. Definition

Assume from now on that, for any $t \in \mathfrak{T}$,

$$0 < m < X(t) < M$$

where m, M are *appreciable*.[8] This is a realistic assumption if $X(t)$ is the price of some financial asset $\mathfrak{A}$. The *logarithmic return*, or *log-return*,[9] of X with respect to some limited time interval $\Delta T > 0$ is the time series $R_{\Delta T}$ defined by

$$R_{\Delta T}(X)(t) = \ln\left(\frac{X(t)}{X(t - \Delta T)}\right) = \ln X(t) - \ln X(t - \Delta T)$$

From $\frac{X(t)}{X(t-\Delta T)} = 1 + \frac{X(t)-X(t-\Delta T)}{X(t-\Delta T)}$, we know that

$$R_{\Delta T}(X)(t) \simeq \frac{X(t) - X(t - \Delta T)}{X(t - \Delta T)} \tag{3}$$

if $X(t) - X(t - \Delta T)$ is infinitesimal. The right handside of Equation (3) is the *arithmetic* return.

The *normalized* logarithmic return is

$$r_{\Delta T}(X)(t) = \frac{R_{\Delta T}(t)}{\Delta T} \tag{4}$$

[2]A real number is *limited* if, and only if, it is not infinitely large.

[3]$a \simeq b$ means that $a - b$ is infinitesimal.

[4]The set R is said to be *rare* [6] if, for any standard real number $\alpha > 0$, there exists an internal set $B \supset A$ such that $m(B) \leq \alpha$.

[5]A set is *quadrable* [6] if its boundary is rare.

[6]Remember that this result led to a new foundation [9] of the analysis of noises in automatic control and in signal processing. A more down to earth exposition may be found in [26].

[7]$E(X)(t)$ was called *trend* in our previous publications [11], [12], [13], [14].

[8]A real number is *appreciable* if, and only if, it is neither infinitely small nor infinitely large.

[9]The terminology *continuously compounded return* is also used. See, *e.g.*, [5] for more details.

B. Mean

B.1 Definition

Replace $X : \mathfrak{T} \to \mathbb{R}$ by

$$\ln X : \mathfrak{T} \to \mathbb{R}, \quad t \mapsto \ln(X(t))$$

where the logarithms of the prices are taken into account. Apply the Cartier-Perrin theorem to $\ln X$. The *mean*, or *average*, of $r_{\Delta T}(t)$ given by Equation (4) is

$$\boxed{\bar{r}_{\Delta T}(X)(t) = \frac{E(\ln X)(t) - E(\ln X)(t - \Delta T)}{\Delta T}} \qquad (5)$$

As a matter of fact $r_{\Delta T}(X)$ and $\bar{r}_{\Delta T}(X)$ are related by

$$r_{\Delta T}(X)(t) = \bar{r}_{\Delta T}(X)(t) + \text{quick fluctuations}$$

Assume that $E(X)$ and $E(\ln X)$ are differentiable according to Section II-B.2. Call the derivative of $E(\ln X)$ the *normalized mean logarithmic instantaneous return* and write

$$\boxed{\bar{r}(X)(t) = \frac{d}{dt}E(\ln X)(t)} \qquad (6)$$

Note that $E(\ln X)(t) \simeq \ln(E(X(t)))$ if in Equation (1) $X_{\text{fluctuation}}(t) \simeq 0$. Then $\bar{r}(X)(t) \simeq \frac{\frac{d}{dt}E(X)(t)}{E(X)(t)}$.

B.2 Application to beta

Take two assets $\mathfrak{A}$ and $\mathfrak{B}$ such that their normalized logarithmic returns $r_{\Delta T}(\mathfrak{A})(t)$ and $r_{\Delta T}(\mathfrak{B})(t)$, defined by Equation (4), exist.[10] Following Equation (5), consider the space curve $t, \bar{r}_{\Delta T}(\mathfrak{A})(t), \bar{r}_{\Delta T}(\mathfrak{B})(t)$ in the Euclidean space with coordinates t, x, y. Its projection on the x, y plane is the plane curve $\mathfrak{C}$ defined by

$$x_{\mathfrak{C}}(t) = \bar{r}_{\Delta T}(\mathfrak{A})(t), y_{\mathfrak{C}}(t) = \bar{r}_{\Delta T}(\mathfrak{B})(t)$$

The tangent of $\mathfrak{C}$ at a regular point, which is defined by $\frac{dx_{\mathfrak{C}}(t)}{dt}, \frac{dy_{\mathfrak{C}}(t)}{dt}$, yields, if $\frac{dx_{\mathfrak{C}}(t)}{dt} \neq 0$,

$$\Delta y_{\mathfrak{C}} \approx \beta(t) \Delta x_{\mathfrak{C}} \qquad (7)$$

where

- $\Delta x_{\mathfrak{C}} = x_{\mathfrak{C}}(t+h) - x_{\mathfrak{C}}(t)$, $\Delta y_{\mathfrak{C}} = y_{\mathfrak{C}}(t+h) - y_{\mathfrak{C}}(t)$;
- $h \in \mathbb{R}$ is "small";
- $$\beta(t) = \frac{\frac{dy_{\mathfrak{C}}(t)}{dt}}{\frac{dx_{\mathfrak{C}}(t)}{dt}} \qquad (8)$$

When $y_{\mathfrak{C}}(t)$ may be viewed locally as a smooth function of $x_{\mathfrak{C}}(t)$, Equation (8) becomes

$$\beta(t) = \frac{dy_{\mathfrak{C}}}{dx_{\mathfrak{C}}}$$

B.3 The Treynor ratio of an asset

Let $\beta_{\mathfrak{M}}(\mathfrak{A})(t)$ be the beta coefficient defined in Section III-B.2 for $\mathfrak{A}$ with respect to the market portfolio $\mathfrak{M}$. Define the *Treynor ratio* and the *instantaneous Treynor ratio* of $\mathfrak{A}$ with respect to $\mathfrak{M}$ respectively by

$$\boxed{\text{TR}_{\mathfrak{M},\Delta T}(\mathfrak{A})(t) = \frac{\bar{r}_{\Delta T}(\mathfrak{A})(t)}{\beta_{\mathfrak{M}}(\mathfrak{A})(t)}, \quad \text{TR}_{\mathfrak{M}}(\mathfrak{A})(t) = \frac{\bar{r}(\mathfrak{A})(t)}{\beta_{\mathfrak{M}}(\mathfrak{A})(t)}}$$

[10]This Section is adapting for returns the presentation in [13].

C. Volatility

Formulae (2), (4), (5), (6) yield the following mathematical definition of the *volatility* of the asset $\mathfrak{A}$:

$$\boxed{\mathbf{vol}_{\Delta T}(\mathfrak{A})(t) = \sqrt{E(r_{\Delta T} - \bar{r}_{\Delta T})^2(t)}} \qquad (9)$$

which yields

$$\boxed{\mathbf{vol}_{\Delta T}(\mathfrak{A})(t) \simeq \sqrt{E(r_{\Delta T}^2)(t) - (\bar{r}_{\Delta T}(t))^2}}$$

The value at time t of $\mathbf{vol}_{\Delta T}(\mathfrak{A})$ may be viewed as the *actual* volatility (see, *e.g.*, [37], chap. 49, pp. 813-814).

Remark 6 : A crucial difference between Formula (9) and the usual *historical*, or *realized*, volatilities (see, *e.g.*, [37], chap. 49, pp. 813-814) lies in the presence of a non-constant mean. It is often assumed to be 0 in the existing literature.

Remark 7 : There is no connection with

- the *implied* volatility, which is connected to the Black-Scholes modeling (see, *e.g.*, [37], chap. 49, pp. 813-814),
- the recent *model-free implied volatility* (see [4], and [23], [29]), although the origin of our viewpoint may be partly traced back to our *model-free control* strategy ([10], [24]).

D. The Sharpe ratio of an asset

Define the *Sharpe ratio* of the asset $\mathfrak{A}$ by

$$\boxed{\text{SR}_{\Delta T}(\mathfrak{A})(t) = \frac{\bar{r}_{\Delta T}(\mathfrak{A})(t)}{\mathbf{vol}_{\Delta T}(\mathfrak{A})(t)}} \qquad (10)$$

According to [1], p. 52, it is quite close to some utilization of the Sharpe ratio in high-frequency trading.

IV. Computer experiments

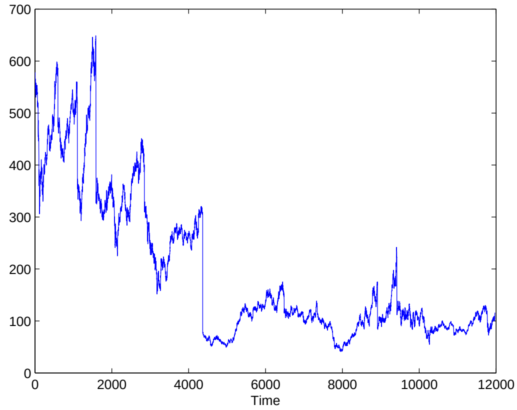
We have utilized the following three listed shares:

1. IBM from 1962-01-02 to 2009-07-21 (11776 days) (Figures 1 and 2),
2. JPMORGAN CHASE (JPM) from 1983-12-30 until 2009-07-21 (6267 days) (Figures 3),
3. COCA COLA (CCE) from 1986-11-24 until 2009-07-21 (5519 days) (Figures 4).

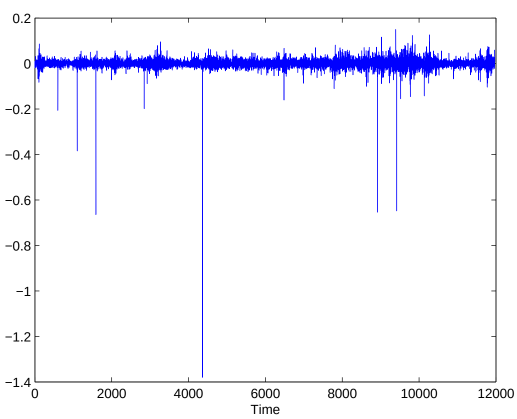
Figures 1 and 3 show a "better" behavior for the normalized mean logarithmic return (6), *i.e.*, $\bar{r}(t)$ is less affected by an abrupt short price variation. Such variations are nevertheless causing important variations on our volatility, with only a "slow mean return". We suggest an adaptive threshold for attenuating this annoying feature, which does not reflect well the price behavior. Note the excellent volatility forecasts which are obtained via elementary numerical recipes as in [11], [12], [13], [14]. Our forecasting results, which are easily computable, seem to be more reliable than those obtained via the celebrated ARCH type techniques, which go back to Engle (see [36] and the references therein).[11]

The beta coefficients is computed with respect to the S&P 500 (see Figures 5). The results displayed in Figures 6 are obtained via the numerical techniques of [13].

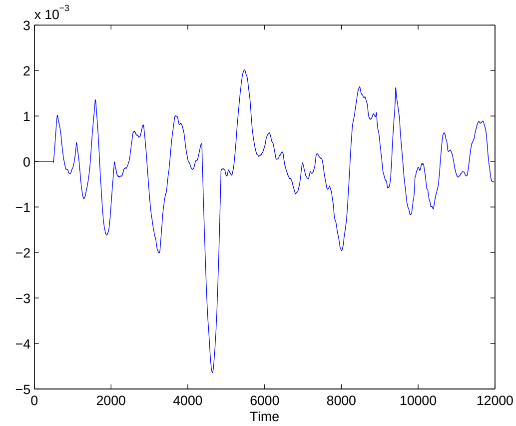
[11]Those comparisons need to be further investigated.



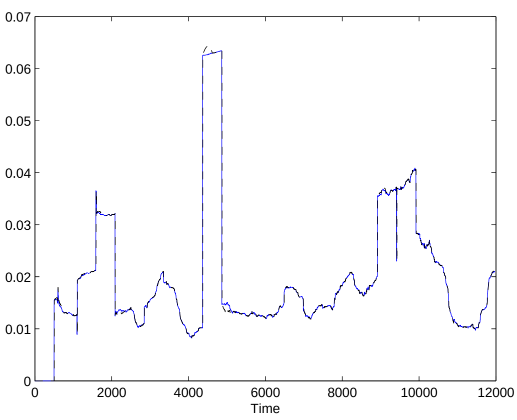
(a)Daily price



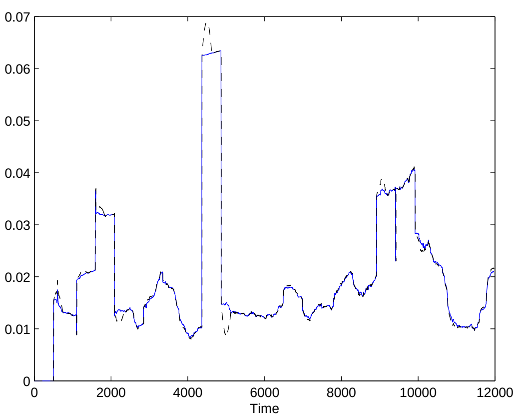
(b)Normalized logarithmic return $r(t)$



(c)Normalized mean logarithmic return $\bar{r}(t)$

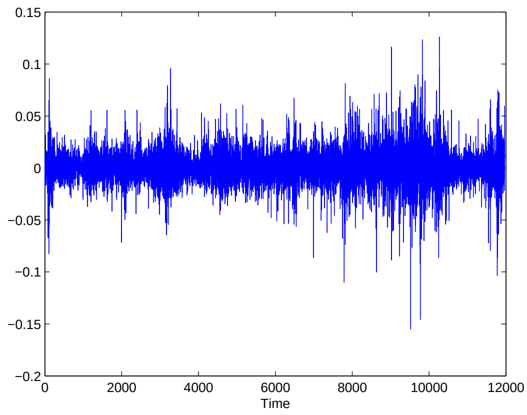


(d)$\mathbf{vol}(IBM)(t)$ (–) and 5 days forecasting (- -)

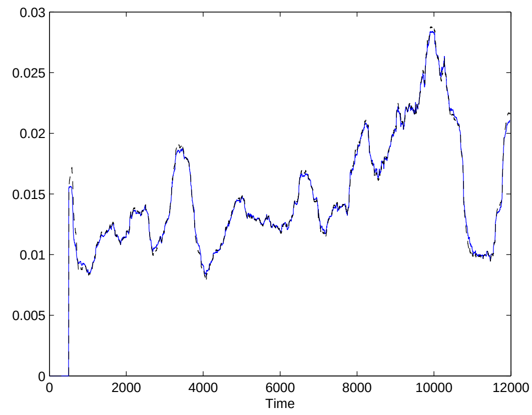


(e)$\mathbf{vol}(IBM)(t)$ (–) and 20 days forecasting (- -)

Fig. 1. IBM

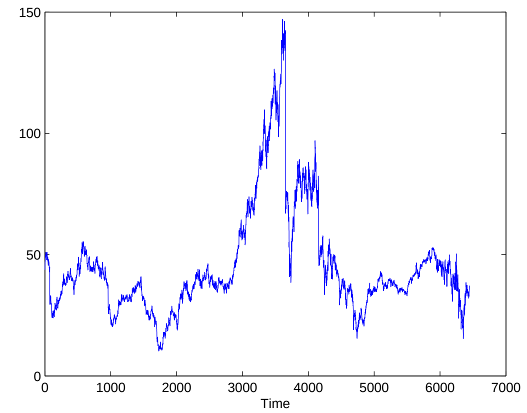


(a)Modified normalized logarithmic return $r(t)$

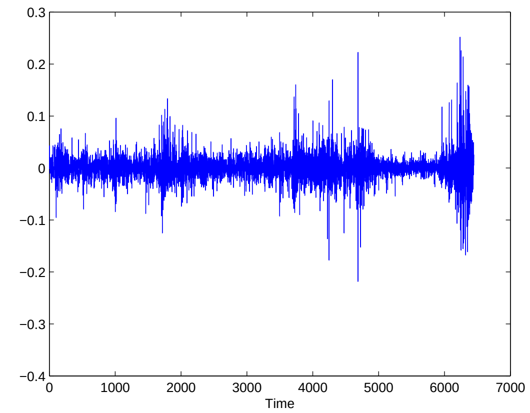


(b)$\mathbf{vol}(IBM)(t)$ (–) and 20 days forecasting (- -)

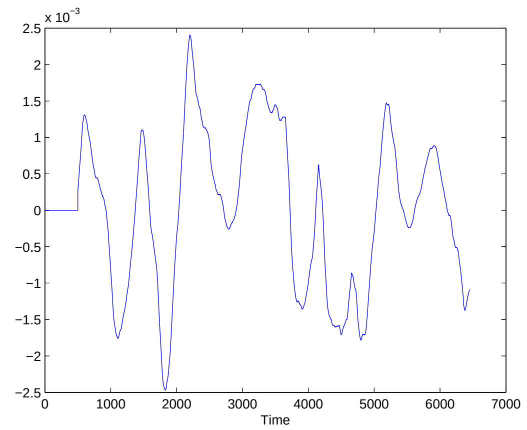
Fig. 2. IBM



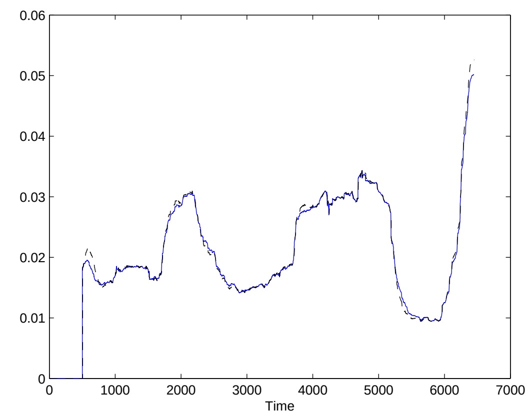
(a)Daily price



(b)Modified normalized logarithmic return $r(t)$

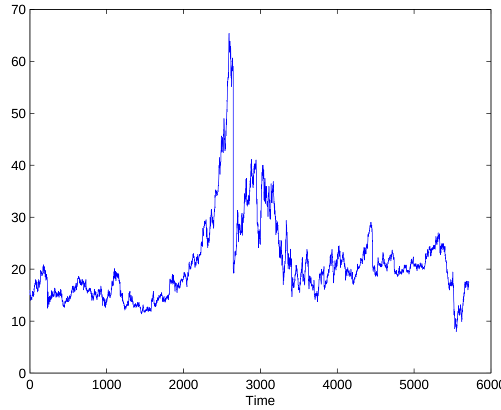


(c)Normalized mean logarithmic return $\bar{r}(t)$

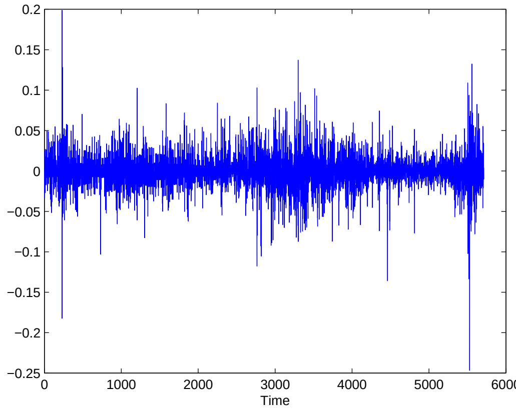


(d)$\mathbf{vol}(JPM)(t)$ (–) and 20 days forecasting (- -)

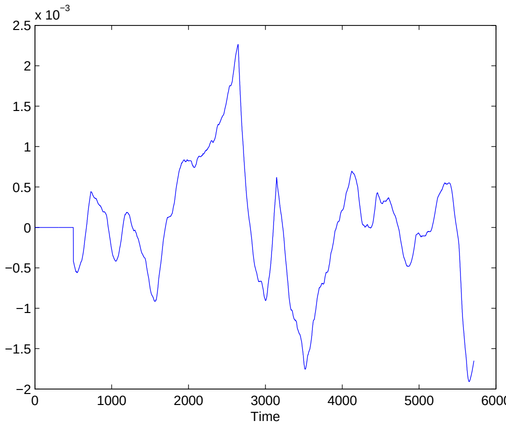
Fig. 3. JPMORGAN CHASE (JPM)



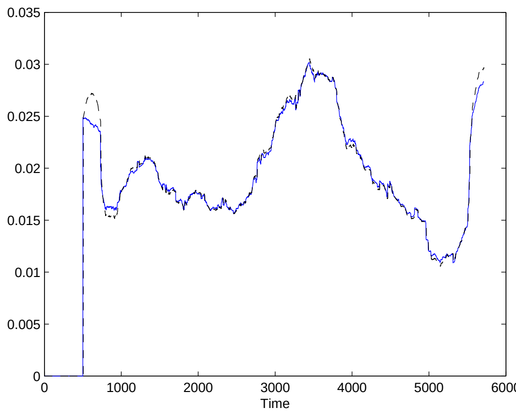
(a)Daily price



(b)Modified normalized logarithmic return $r(t)$



(c)Normalized mean logarithmic return $\bar{r}(t)$



(d)$\mathbf{vol}(CCE)(t)$ (–) and 20 days forecasting (- -)

Fig. 4. COCA COLA (CCE)

Figure 7 displays the Sharpe ratio of S&P 500. With $\Delta t = 10$ a trend is difficult to guess in Figure 7-(a). Figure 7-(b) on the other hand, where $\Delta t = 100$, exhibits a well-defined trend which yields a quite accurate forecasting of 10 days.

V. Conclusion

Although we have proposed a precise and elegant mathematical definition of volatility, which

- yields efficient and easily implementable computations,
- will soon be exploited for a dynamic portfolio management [15],

the harsh criticisms against its importance in financial engineering should certainly not be dismissed (see, *e.g.*, [33]). Note for instance that we have not tried here to forecast extreme events, *i.e.*, abrupt changes (see [16]) with this tool. This aim has been already quite successfully achieved in [11], [12], [13], [14], not via volatility but by taking advantage of indicators that are related to prices and not to returns.

References

[1] Alridge I., *High-Frequency Trading*. Wiley, 2010.

[2] Béchu T., Bertrand E., Nebenzahl J., *L'analyse technique* (6^e éd.). Economica, 2008.

[3] Bodie Z., Kane A., Marcus A.J., *Investments* (7^{th} ed.). McGraw-Hill, 2008.

[4] Britten-Jones M., Neuberger A., Option prices, implied price processes, and stochastic volatility. *J. Finance*, vol. 55, pp. 839-866, 2000.

[5] Campbell J.Y., Lo A.W., MacKinlay A.C., *The Econometrics of Financial Markets*. Princeton University Press, 1997.

[6] Cartier P., Perrin Y., Integration over finite sets. In *Nonstandard Analysis in Practice*, F. & M. Diener (Eds), Springer, 1995, pp. 195-204.

[7] Diener F., Diener M., Tutorial. In F. & M. Diener (Eds): *Nonstandard Analysis in Practice*. Springer, pp. 1-21, 1995.

[8] Diener F., Reeb G., *Analyse non standard*. Hermann, 1989.

[9] Fliess M., Analyse non standard du bruit. *C.R. Acad. Sci. Paris Ser. I*, vol. 342, pp. 797-802, 2006.

[10] Fliess M., Join C., Commande sans modèle et commande à modèle restreint. *e-STA*, vol. 5 (n° 4), pp. 1-23, 2008 (available at `http://hal.archives-ouvertes.fr/inria-00288107/en/`).

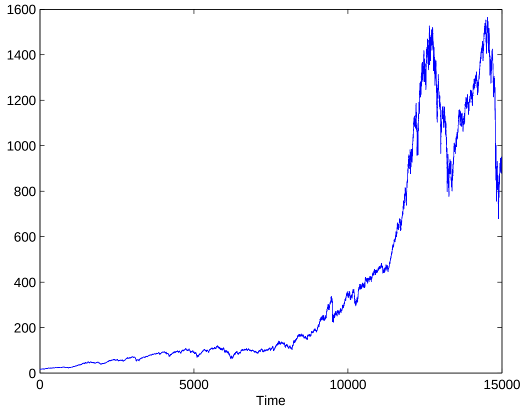
[11] Fliess M., Join C., A mathematical proof of the existence of trends in financial time series. In *Systems Theory: Modeling, Analysis and Control*, A. El Jai, L. Afifi, E. Zerrik (Eds), Presses Universitaires de Perpignan, 2009, pp. 43–62 (available at `http://hal.archives-ouvertes.fr/inria-00352834/en/`).

[12] Fliess M., Join C., Towards new technical indicators for trading systems and risk management. 15^{th} IFAC Symp. System Identif., Saint-Malo, 2009 (available at `http://hal.archives-ouvertes.fr/inria-00370168/en/`).

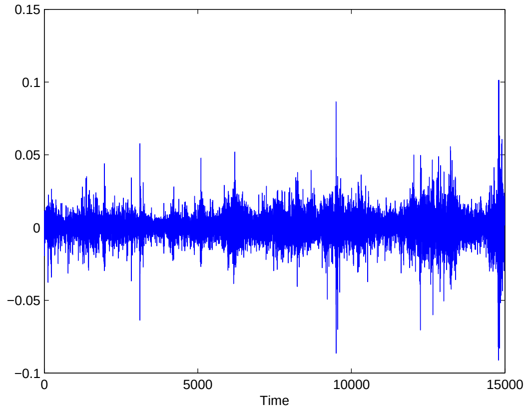
[13] Fliess M., Join C., Systematic risk analysis: first steps towards a new definition of beta. COGIS, Paris, 2009 (available at `http://hal.archives-ouvertes.fr/inria-00425077/en/`).

[14] Fliess M., Join C., Delta hedging in financial engineering: towards a model-free setting. 18^{th} Medit. Conf. Control Automat., Marrakech, 2010 (available at `http://hal.archives-ouvertes.fr/inria-00479824/en/`).

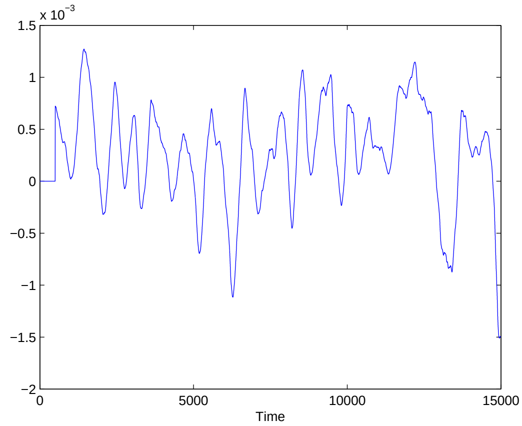
[15] Fliess M., Join C., Hatt. F., A-t-on vraiment besoin de modèles probabilistes en ingénierie financière ?. Conf. médit. ingénierie



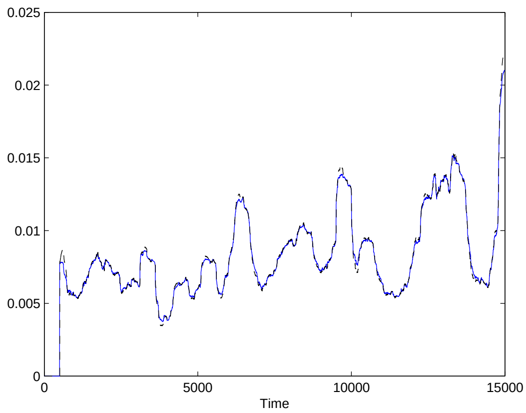
(a)Daily price



(b)Modified normalized logarithmic return $r(t)$



(c)Normalized mean logarithmic return $\bar{r}(t)$



(d)**vol**$(S\&P500)(t)$ (–) and 20 days forecasting (- -)

Fig. 5. S&P 500

sûre systèmes complexes, Agadir, 2011 (soon available at `http://hal.archives-ouvertes.fr/`).

[16] Fliess M., Join C., Mboup M., Algebraic change-point detection. *Applicable Algebra Engin. Communic. Comput.*, vol. 21, pp. 131-143, 2010.

[17] Fliess M., Join C., Sira-Ramírez H., Non-linear estimation is easy. *Int. J. Model. Identif. Control*, vol. 4, pp. 12-27, 2008 (available at `http://hal.archives-ouvertes.fr/inria-00158855/en/`).

[18] Franke J., Härdle W.K., Hafner C.M., *Statistics of Financial Markets* (2^{nd} ed.). Springer, 2008.

[19] García Collado F.A., d'Andréa-Novel B., Fliess M., Mounier H., Analyse fréquentielle des dérivateurs algébriques. XXII^e Coll. GRETSI, Dijon, 2009 (available at `http://hal.archives-ouvertes.fr/inria-00394972/en/`).

[20] Goldstein D.G., Taleb N.N., We don't quite know what we are talking about when we talk about volatility. *J. Portfolio Management*, vol. 33, pp. 84-86, 2007.

[21] Gunn M., *Trading Regime Analysis*. Wiley, 2009.

[22] Hull J.C., *Options, Futures, and Other Derivatives* (7^{th} ed.). Prentice Hall, 2007.

[23] Jiang G.J., Tian Y.S., The model-free implied volatility and its information content. *Rev. Financial Studies*, vol. 18, pp. 1305-1342, 2005.

[24] Join C., Robert G., Fliess M., Vers une commande sans modèle pour aménagements hydroélectriques en cascade. 6^e Conf. Internat. Francoph. Automat., Nancy, 2010 (available at `http://hal.archives-ouvertes.fr/inria-00460912/en/`).

[25] Kirkpatrick C.D., Dahlquist J.R., *Technical Analysis: The Complete Resource for Financial Market Technicians* (2^{nd} ed.). FT Press, 2010.

[26] Lobry C., Sari T., Nonstandard analysis and representation of reality. *Int. J. Control*, vol. 81, pp. 517-534, 2008.

[27] Mboup M., Join C., Fliess M., Numerical differentiation with annihilators in noisy environment. *Numer. Algor.*, vol. 50, pp. 439-467, 2009.

[28] Roncalli T., *La gestion d'actifs quantitative*. Economica, 2010.

[29] Rouah F.D., Vainberg G., *Option Pricing Models and Volatility*. Wiley, 2007.

[30] Sharpe W.F., Mutual fund performance. *J. Business*, vol. 39, pp. 119-138, 1966.

[31] Sharpe W.F., The Sharpe ratio. *J. Portfolio Management*, vol. 21, pp. 49-58, 1994.

[32] Sinclair E., *Volatility Trading*. Wiley, 2008.

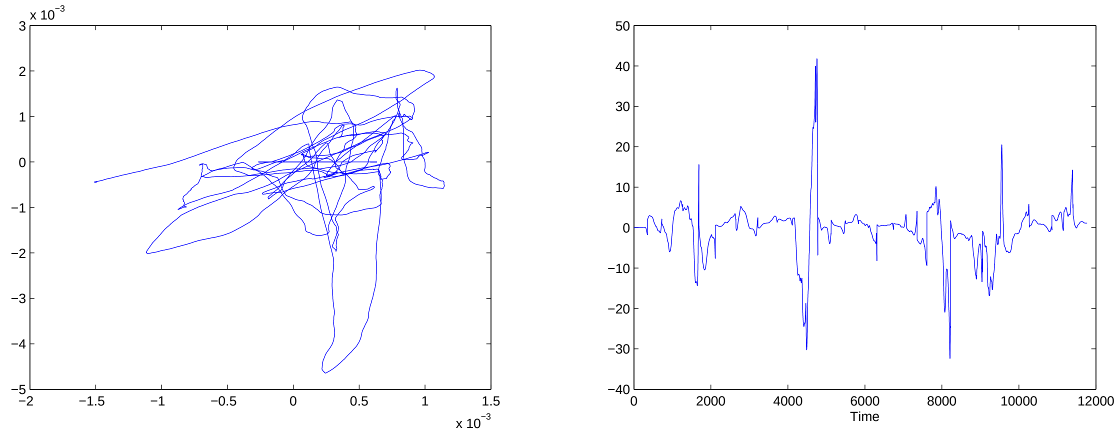
[33] Taleb N.N., Errors, robustness, and the fourth quadrant. *Int. J. Forecasting*, vol. 25, pp. 744-759, 2009.

[34] Treynor J.L., *Treynor on Institutional Investing*. Wiley, 2008.

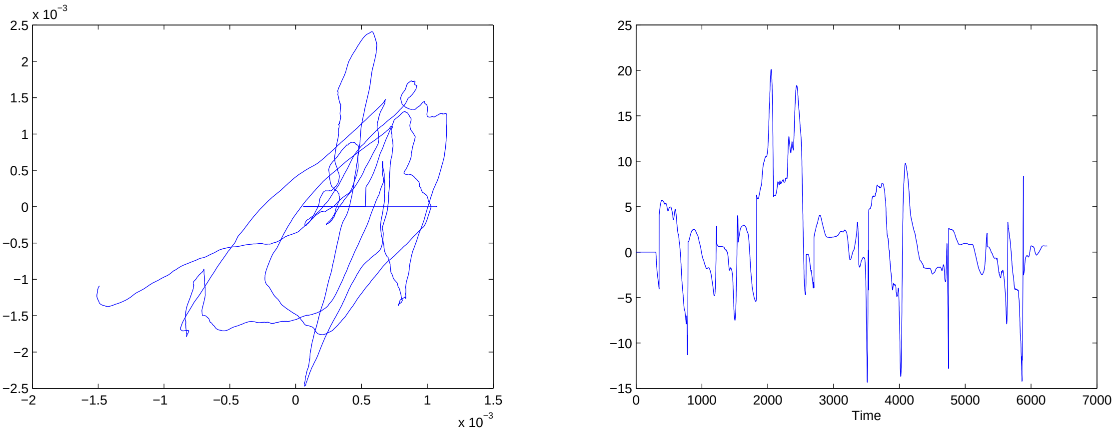
[35] Tsay R.S., *Analysis of Financial Time Series* (2^{nd} ed.). Wiley, 2005.

[36] Watson M., Bollerslev T., Russel J. (Eds), *Volatility and Time Series Econometrics – Essays in Honor of Robert Engle*. Oxford University Press, 2010.

[37] Wilmott P., *Paul Wilmott on Quantitative Finance*, 3 vol. (2^{nd} ed.). Wiley, 2006.

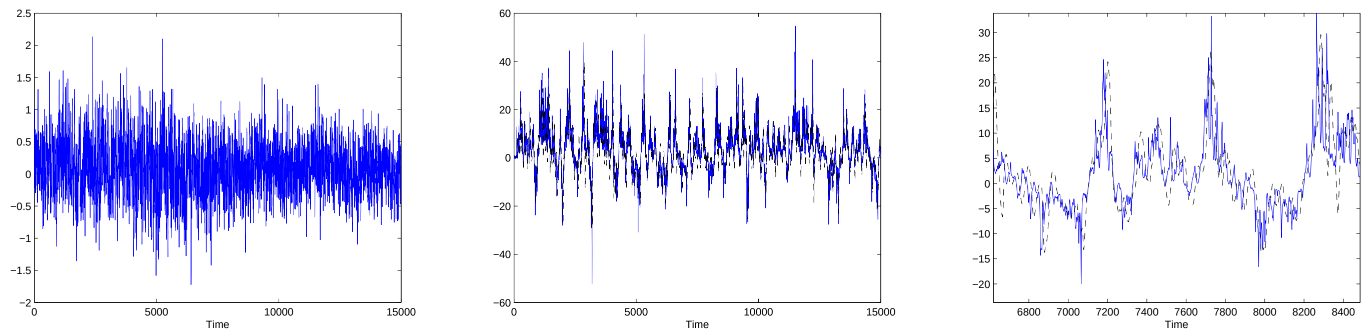


(a)$\mathfrak{C}$: $\bar{r}_{IBM}(\bar{r}_{S\&P500})$ (b)IBM's $\beta(t)$



(c)$\mathfrak{C}$: $\bar{r}_{JPM}(\bar{r}_{S\&P500})$ (d)JPM's $\beta(t)$

Fig. 6.



(a)SR$_{10}(S\&P500)$ (b)SR$_{100}(S\&P500)$ (–) and 10 days forecasting (- -) (c)Zoom of 7-(b)

Fig. 7.