Unusual stability of a one-parameter family of dissipative solitons due to spectral filtering and nonlinearity saturation

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Stability of one-parameter family of dissipative solitons seen in the cubic-quintic complex Ginzburg-Landau equation is studied. It is found that an unusually strong stability occurs for solitons controlled by the spectral filtering and nonlinearity saturation simultaneously, consistently with the linear stability analysis and confirmed by large-perturbation numerical simulations. Two universal types of bifurcations in the spectrum structure are demonstrated.

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Studies of spatial and temporal complexity in nonlinear systems bring a plethora of fundamental concepts. Among the most fascinating ones is the soliton, a special type of localized wave packets that does not broaden while propagating in a dispersive environment, ever observed in areas ranging from hydrodynamics, biology, nonlinear optics, to Bose-Einstein condensates [1]. Originally, soliton is a terminology reserved for the integrable nonlinear differential equations which have one feature in common: they are all conservative and are thus derivable from a Hamiltonian [2, 3]. However, current state of knowledge on nonlinear science has significantly extended the versatility of soliton concept to as far as general dissipative systems: hence coined the name "dissipative soliton" [4, 5].

In the context of the complex Ginzburg-Landau (CGL) equations, one can identify the dissipative solitons as the fixed points (stationary solitons) or the limit cycles (pulsating solitons) in phase space [6–8]. Hence, the dynamics of dissipative solitons always depends drastically on system parameters and behaves as strong nonlinear attractors [9, 10], analogous to the attracting behavior of optical similaritons in amplified nonlinear materials [11].

Intriguingly, as was advanced by Akhmediev and Afanasjev, the CGL equation also admits a class of dissipative solitons with *arbitrary amplitude* by particular choice of system parameters [12]. For later convenience, we dub them as "Akhmediev-Afanasjev (AA) solitons". Such one-parameter solitons can be reminiscent of the analogs in the cubic-quintic nonlinear Schrödinger (CQNLS) equation [1]. It is well known that the latter ones may possess either internal modes responsible for the long-lived periodic oscillations of amplitude [13] or unstable modes resulting in a soliton collapse or a decay into linear dispersive waves [14]. Hence, a natural question arises as to whether the AA solitons here can invalidate these instability rules and if so, what mechanism is responsible for this.

In this Rapid Communication, we attempt to address this issue based on the linear stability analysis along with extensive numerical simulations. In contrast to what might be naively expected, the AA solitons are found to exhibit unusual stability as the spectral filtering and saturable nonlinearity are taken into account simultaneously: they can recover cleanly from arbitrary, large perturbations. Contrarily, those in the positive quintic nonlinearity are either completely unstable or much less stable against perturbations. In addition, bifurcations from continuous spectra and neutral modes are highlighted in interpretation of soliton unusual stability.

For our studies, we consider the cubic-quintic CGL equation as [8–10, 12]

$$i\psi_z + \left(\frac{\mathcal{D}}{2} - i\beta\right)\psi_{\tau\tau} + (1 - i\epsilon)\,|\psi|^2\psi + (\nu - i\mu)\,|\psi|^4\psi = 0, \tag{1}$$

where ψ is the normalized envelope of the field, z is the distance that the pulse travels, τ is the retarded time, $\mathcal{D}$ is the group-velocity dispersion coefficient, β describes the spectral filtering ($\beta > 0$), ϵ accounts for the nonlinear gain, and ν and μ are responsible for the quintic saturations or corrections to the cubic nonlinearity. In our context, additional two parametric relations $\epsilon = \frac{2\beta}{3\sigma+\mathcal{D}}$ and $\mu = \frac{2\beta\nu}{2\sigma+\mathcal{D}}$ (here $\sigma = \sqrt{4\beta^2 + \mathcal{D}^2}$) are assumed, which reduce Eq. (1) to be actually dependent on three independent parameters, say, β, ν, and $\mathcal{D}$. In cases of anomalous, normal, and zero dispersions, $\mathcal{D}$ may be normalized to be 1, -1, or 0.

Under the circumstances, Eq. (1) allows exact one-parameter family of soliton solutions written as [12]

$$\psi(z,\tau) = U_0(\tau)e^{i\Omega z}, \tag{2}$$

$$U_0(\tau) = \left[\frac{P}{1 + (1-\eta)\sinh^2\left(\frac{\tau-\tau_0}{T}\right)}\right]^{\frac{1+id}{2}}, \tag{3}$$

where $\Omega = 2\sigma P/(3\sigma + \mathcal{D}) + \nu\sigma P^2/(2\sigma + \mathcal{D})$, $\eta = -\nu P/\left[\nu P + (4\sigma + 2\mathcal{D})/(3\sigma + \mathcal{D})\right]$, $d = (\sigma - \mathcal{D})/2\beta$, and $T = (\sigma/2\beta)\sqrt{(\sigma - \mathcal{D})/\Omega}$. Here P is the peak power treated as one free parameter. τ_0 is a central pulse position and can usually be set to zero for symmetric input.

This solution, termed AA soliton, can be applied either to (i) the positive quintic nonlinearity case ($\nu > 0$) for which $-1 < \eta \leqslant 0$ and P is unbounded above, or to (*ii*) the saturable case ($\nu < 0$) for which $0 \leqslant \eta < 1$ and $0 \leqslant P < P_m = -\frac{2\sigma+\mathcal{D}}{\nu(3\sigma+\mathcal{D})}$. In both cases, the soliton energy $E = \int_{-\infty}^{\infty} |\psi(z,\tau)|^2 d\tau$ can be given by

$$E = \sqrt{\frac{\sigma(2\sigma+\mathcal{D})(\sigma-\mathcal{D})}{-\nu\beta^2}}\operatorname{arctanh}(\sqrt{\eta}). \quad (4)$$

Clearly, for $\nu < 0$, the energy increases nonlinearly with P and explodes in the limit $P \to P_m$. This feature can be used to generate highly energetic optical pulses seeded for filamentation in diverse transparent media [15].

It is noteworthy that as $\beta = 0$, Eq. (1) can be transformed into the CQNLS equation which also has exact one-parameter family of soliton solutions given by the ansatz (2) and (3), but with $\Omega = P(2\nu P + 3)/6$, $\eta = -2\nu P/(2\nu P+3)$, $d = 0$, and $T = \sqrt{3\mathcal{D}/(2\nu P^2 + 3P)}$. Further, if ν equals to zero, Eq. (1) then reduces to the NLS equation and the bright soliton solution ($\mathcal{D} = 1$) is defined by $\Omega = P/2$, $T = \sqrt{1/P}$, and $\eta = d = 0$ (zero soliton velocity is assumed). As two typical Hamiltanian systems, the soliton stability can be determined by a simple rule known as Vakhitov-Kolokolov (VK) criterion [1]. This criterion states that a soliton is stable only if $dE/d\Omega > 0$ [16]. However, for our dissipative soliton system, it fails to apply as there occurs a complex conjugate pair of eigenvalues beyond the VK description [1]. Hence, in order to reveal the stability features of AA solitons, a powerful linear stability analysis is needed [5].

We consider the evolution of a small perturbation of the soliton by modifying the soliton solution (2) as $\psi(z,\tau) = [U_0(\tau) + f(\tau)\exp(-i\lambda z) + g^*(\tau)\exp(i\lambda^* z)]\exp(i\Omega z)$, where $f(\tau)$ and $g(\tau)$ are perturbation functions, λ is a complex number, and the asterisk denotes the complex conjugation. Substituting this perturbed solution into Eq. (1) and linearizing in f and g, we obtain

$$\mathbf{L}\Psi = \lambda\Psi, \qquad \Psi = \begin{pmatrix} f \\ g \end{pmatrix}, \quad (5)$$

where $\mathbf{L}$ is a linear operator given by

$$\mathbf{L} = \begin{pmatrix} -\left(\frac{\mathcal{D}}{2} - i\beta\right)\frac{d^2}{d\tau^2} + \wp & -\aleph \\ \aleph^* & \left(\frac{\mathcal{D}}{2} + i\beta\right)\frac{d^2}{d\tau^2} - \wp^* \end{pmatrix} \quad (6)$$

with $\wp = \Omega - 2(1-i\epsilon)|U_0|^2 - 3(\nu - i\mu)|U_0|^4$ and $\aleph = (1-i\epsilon)U_0^2 + 2(\nu - i\mu)|U_0|^2 U_0^2$.

An inspection of the above eigenvalue problem reveals that, if (λ, f, g) is a solution, then so is $(-\lambda^*, g^*, f^*)$. This implies the spectrum relation here to be $\lambda \to -\lambda^*$, different from the Hamiltonian situation in which the spectra are usually related by $\lambda \to \pm\lambda^*$ [1], and thereby any eigenvalue with $\mathrm{Im}(\lambda) \neq 0$ popping out suggests an instability of soliton. As will be shown below, our eigenvalue problem involves eigenvalues all with $\mathrm{Im}(\lambda) \leqslant 0$ for $\nu < 0$. This indicates a strong linear stability of the soliton (2) due to an exponential decay of dispersive modes.

We begin with the essential (continuous) spectra and, by examining the asymptotic operator $\mathbf{L}|_{\tau\to\infty}$, find them to be $\lambda_c^{\pm} = \pm[\Omega + (\mathcal{D}/2)k^2] - i\beta k^2$ (here k is arbitrarily real), symmetrically located in the λ-plane. Trivially, zero is a discrete eigenvalue of geometric multiplicity 2, with two localized neutral modes as

$$\Psi_e = \begin{pmatrix} U_0 \\ -U_0^* \end{pmatrix}, \quad \Psi_o = \begin{pmatrix} \frac{d}{d\tau}U_0 \\ \frac{d}{d\tau}U_0^* \end{pmatrix}, \quad (7)$$

which correspond to the translational and phase invariances of soliton, respectively. The other nontrivial discrete eigenvalues depend on the parameters β, ν, $\mathcal{D}$, and P. To find them, we perform direct numerical calculations of Eq. (5) with error $\Delta\lambda < 10^{-12}$, based on a modified squared-operator Euler iteration scheme [17] which was shown to converge to any discrete eigenvalue in the stability spectrum under mild conditions.

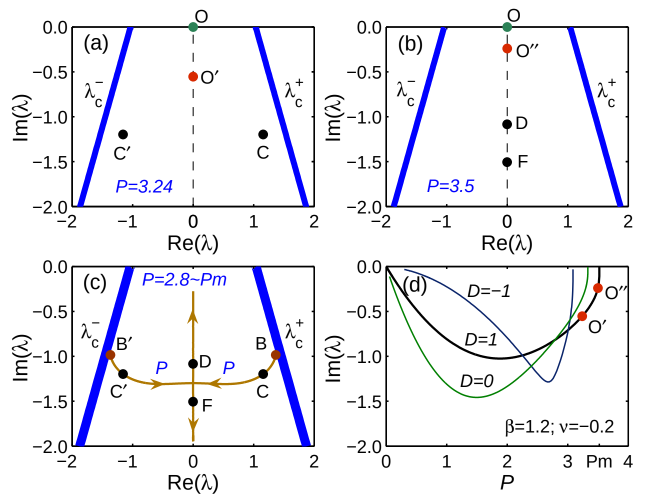

FIG. 1. (color online) Spectrum structure for an AA soliton at $\beta = 1.2$, $\nu = -0.2$, and $\mathcal{D} = 1$: (a) $P = 3.24$; (b) $P = 3.5$, with colored circle and blue shaded region denoting the discrete spectrum and continuum, respectively. In (c) a complex conjugate pair of eigenvalues evolves towards two purely imaginary ones as P grows from 2.8 to $P_m = 155/44$, and in (d) purely imaginary eigenvalues bifurcate from neutral modes for anomalous, zero, and normal dispersions, respectively.

The case of saturable nonlinearity ($\nu < 0$) is our first concern. Figures 1(a) and 1(b) illustrate, respectively, the spectra of an AA soliton in the anomalous dispersion for different peak powers. Both panels involve four discrete eigenvalues (colored circles) with $\mathrm{Im}(\lambda) \leqslant 0$, confined between two continuous branches λ_c^- and λ_c^+ (blue shaded). The main discrepancy lies in that in panel (a) there appears a complex conjugate pair of eigenvalues ($\lambda_{C,C'} = \pm 1.1586 - 1.1972i$), while in panel (b) two purely imaginary eigenvalues ($\lambda_D = -1.0839i$, $\lambda_F = -1.5047i$) exist instead. As indicated in Fig. 1(c), this pair of complex eigenvalues indeed bifurcates from the continuous parts as

$P = 2.8$ (red circles), and then evolves into two purely imaginary ones as P grows larger than 3.498. In contrast, another purely imaginary eigenvalue in panel (a) or (b) ($\lambda_{O'} = -0.5535i$, $\lambda_{O''} = -0.2404i$) bifurcates from the neutral mode spectrum O at $P = 0$, see the black curve in Fig. 1(d). Accordingly, we term the former process as *continuous spectrum bifurcation* and the latter one as *neutral mode bifurcation*, both of which are universal in the analysis of the stability of complex soliton systems [13, 18].

The same bifurcation processes also occur in the case of normal or zero dispersion. Here we only demonstrate in Fig. 1(d) the neutral mode bifurcations, in order to compare them under different dispersions. It should be emphasized that in the case of normal dispersion, there appears a bifurcation from the even neutral mode Ψ_e, different from those in the anomalous and zero dispersions which bifurcate from the odd neutral mode Ψ_o, see Eq. (7). Besides, we point out that in the close vicinity of $P \to P_m$, there still exist higher-order eigenmodes (more than one node in their amplitude) which involve negatively imaginary eigenvalues as well. For brevity, we do not plot them in Fig. 1(d).

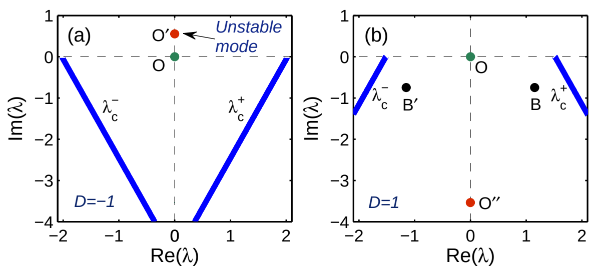

FIG. 2. (color online) Spectrum structures for AA solitons calculated at $\beta = 1.2$, $\nu = 0.2$, and $P = 2$: (a) normal dispersion ($\mathcal{D} = -1$); (b) anomalous dispersion ($\mathcal{D} = 1$).

As regards the case of $\nu > 0$, calculations on spectrum structures show that AA solitons are always unstable in the normal dispersion because of a positively imaginary eigenvalue [see the arrow indication in Fig. 2(a)], while those in the anomalous dispersion are linearly stable, as confirmed numerically in Ref. [19]. For these two types of AA solitons, there still occur the neutral mode and/or continuous spectrum bifurcations, as seen in the case of $\nu < 0$. For instance, the unstable mode ($\lambda_{O'} = 0.5536i$) in Fig. 2(a) and the stable mode ($\lambda_{O''} = -3.5339i$) in Fig. 2(b) are just the ones that bifurcate from the neutral modes O. Of special note is that the continua in Fig. 2(a) intersect (not shown explicitly) in the lower half-plane and no other discrete eigenvalues lie in between. For this case, if ν becomes smaller than zero, the eigenmode O' moves down below O and becomes stable.

The uniqueness of the spectrum structure of AA solitons is evident in comparison with those for NLS and CQNLS solitons. As is well known, the NLS solitons have two degenerate neutral modes of eigenvalue zero [18] [see

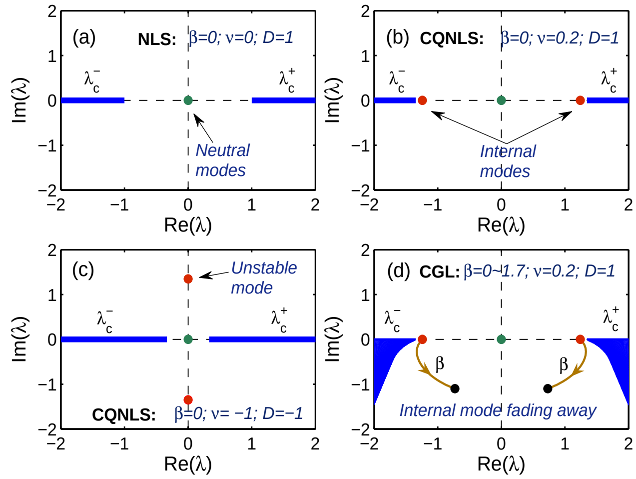

FIG. 3. (color online) Spectrum structures calculated at $P = 2$ for (a) bright NLS soliton, (b) oscillating CQNLS soliton, (c) unstable CQNLS soliton with $\nu < 0$, and (d) CGL (AA) soliton with $\nu > 0$ and β ranging from 0 to 1.7.

Fig. 3(a)], while the CQNLS ones, apart from the neutral modes, can possess either internal modes [13] [see Fig. 3(b)] or an unstable mode [17] [see Fig. 3(c)], depending on what kind of interplay between dispersion and nonlinearity. Obviously, in the light of spectrum complexity in Figs. 1 and 2(b), it is possible that un unusual stability might occur for AA solitons, either with $\nu < 0$ or with $\nu, \mathcal{D} > 0$. To show this, we plot in Fig. 3(d) the fade-away of internal modes as β grows away from zero. Hence in principle, AA solitons do not display a distinct breather feature, unless they are initially perturbed [14].

As neutral modes are unavoidable for all the above solitons (see Figs. 1, 2 and 3), one may ask whether AA solitons, either saturable or for the $\nu, \mathcal{D} > 0$ case, are as *neutrally stable* as the NLS or CQNLS solitons. To answer it, we perform direct numerical simulations of Eq. (1) using the split-step Fourier code. We first consider in Fig. 4 the propagation of four types of solitons perturbed initially by a weak dispersive field nearby. It is clearly seen that two types of Hamiltonian solitons suffer from strong fluctuations in both their soliton part and background [see panels (a) and (b)], whereas AA solitons can recover very cleanly from such perturbation [see panels (c) and (d)]. We argue that it is the effect of spectral filtering that suppresses the instabilities induced by the dispersive field, as revealed in our linear stability analysis above. Additionally, we consider perturbations such as $U_0 \exp(i0.5\tau)$ and find that only the solitons in Figs. 4(c) and 4(d) are in no motion, apart from being translated to a new central position τ_0. This is also a typical feature of beyond neutral stability.

Further simulations show that the AA solitons for the $\nu, \mathcal{D} > 0$ case are not always stable against large bump perturbations, especially for a large β or P. They tend

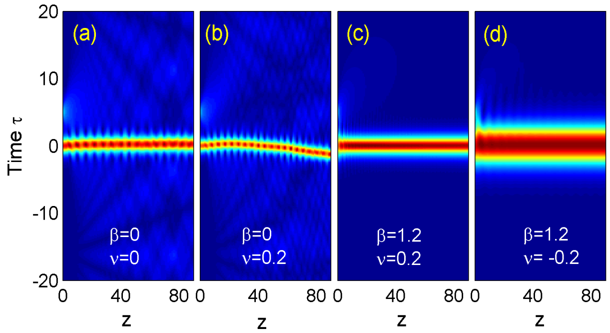

FIG. 4. (color online) Propagation of solitons $U_0(\tau)$ under the same perturbation of a weak dispersive field nearby for $\mathcal{D} = 1$ and $P = 2$: (a) bright NLS soliton; (b) CQNLS soliton; (c) AA soliton with $\nu > 0$; and (d) AA soliton with $\nu < 0$.

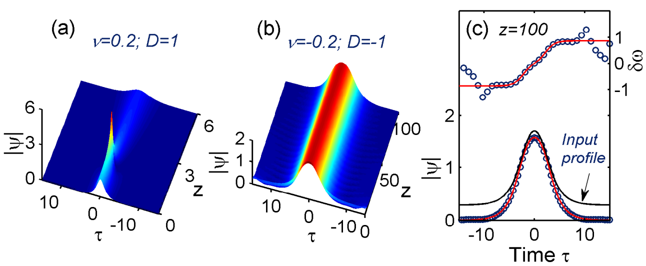

FIG. 5. (color online) Evolution of a perturbed AA soliton ($\beta = 1.0$) for (a) the $\nu, \mathcal{D} > 0$ case and (b) the $\nu < 0$ case. (c) compares the simulations (open circles) of amplitude and chirp in (b) with analytical results (solid curves).

to collapse at the very start and then decay due to a sudden increase of the spectral width, see Fig. 5(a). As one can verify, this collapse-and-arrest process can be qualitatively explained via the rate functional $\frac{d}{dz}E(z) = 4\beta\int_{-\infty}^{\infty}\left(\frac{|\psi|^4}{3\sigma+\mathcal{D}}+\frac{\nu|\psi|^6}{2\sigma+\mathcal{D}}-\frac{1}{2}|\psi_\tau|^2\right)d\tau$ [20]. Thereafter, if the intensity is not high enough to build a new balance for a soliton, the pulses eventually decay to nil. However, those for the saturable case are totally stable for any value of the parameters involved, independently of what kinds of perturbations imposed. Figures. 5(b) and 5(c) show a survival of such an AA soliton from an input pulse of which the amplitude does not localize (nonlocality of 20% peak amplitude), with its profile and chirp $\delta\omega$ excellently consistent with the analytical results. Obviously, this self-healing feature is unaccessible for any neutrally stable soliton seen in Hamiltonian systems [1].

We finally point out that if the parameters ϵ and μ in Eq. (1) are *slightly* changed but following the inequality $(\epsilon - \frac{2\beta}{3\sigma+\mathcal{D}})(\mu - \frac{2\beta\nu}{2\sigma+\mathcal{D}}) < 0$ (here $\beta > 0$, $\nu < 0$), then the AA solitons will move into their fixed-amplitude form but still described very accurately by Eqs. (2) and (3) (fixed point attractors) [10]; otherwise they evolve into fronts or decay to nil. It is suggested that this special type of solitons has an experimental feasibility via tuning reversely the soliton from hard excitation to soft excitation [12].

In conclusion, we investigate the stability of AA solitons using the linear stability analysis and simulations. An unusual stability is expected for AA solitons in the saturable case, manifested by a clean soliton survival from arbitrary, large perturbations (even including large amplitude nonlocalization). From a practical perspective, this finding suggests a new route to the design of hyperstable fiber lasers, mainly owing to the attractor-like stability of these solitons as well as their ability to be softly excited [12]. Moreover, there is another significance in that the spectra of general types of dissipative solitons can be sought by virtue of the perturbation to our dissipative model (1), rather than to the CQNLS equation or to the NLS equation [18].

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[1] Yu.S. Kivshar and G.P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic, San Diego, 2003).
[2] N. Akhmediev and A. Ankiewicz, *Solitons, Nonlinear Pulses and Beams* (Chapman and Hall, London, 1997).
[3] H.A. Haus and W.S. Wong, Rev. Mod. Phys. **68**, 423 (1996).
[4] O. Thual and S. Fauve, J. Phys. (Paris) **49,** 1829 (1988); H. R. Brand and R.J. Deissler, Phys. Rev. Lett. **63**, 2801 (1989).
[5] *Dissipative Solitons*, edited by N. Akhmediev and A. Ankiewicz (Springer, New York, 2005).
[6] W. van Saarloos and P.C. Hohenberg, Physica D **56**, 303 (1992).
[7] I.S. Aranson and L. Kramer, Rev. Mod. Phys. **74**, 99 (2002).
[8] J.M. Soto-Crespo, N. Akhmediev, and A. Ankiewicz, Phys. Rev. Lett. **85**, 2937 (2000).
[9] J.M. Soto-Crespo and N. Akhmediev, Phys. Rev. Lett. **95**, 024101 (2005).
[10] S. Chen, Phys. Rev. E **78,** 025601(R) (2008).
[11] J.M. Dudley *et al.*, Nature Phys. **3**, 597 (2007); S. Chen and J.M. Dudley, Phys. Rev. Lett. **102**, 233903 (2009) .
[12] N. Akhmediev and V.V. Afanasjev, Phys. Rev. Lett. **75**, 2320 (1995).
[13] D.E. Pelinovsky, Yu.S. Kivshar, and V.V. Afanasjev, Physica D **116**, 121 (1998); Yu.S. Kivshar, D.E. Pelinovsky, T. Cretegny, and M. Peyrard, Phys. Rev. Lett. **80**, 5032 (1998).
[14] D.E. Pelinovsky, V.V. Afanasjev, and Yu.S. Kivshar, Phys. Rev. E **53**, 1940 (1996).
[15] A. Couairon and A. Mysyrowicz, Phys. Rep. **441**, 47 (2007).
[16] Y. Sivan, G. Fibich, and M.I. Weinstein, Phys. Rev. Lett. **97**, 193902 (2006).
[17] J. Yang, J. Comput. Phys. **227**, 6862 (2008).
[18] T. Kapitula and B. Sandstede, Physica D **124**, 58 (1998).
[19] J.M. Soto-Crespo, N.N. Akhmediev, V.V. Afanasjev, and S. Wabnitz, Phys. Rev. E **55**, 4783 (1997).
[20] N.N. Akhmediev, A. Ankiewicz, and J.M. Soto-Crespo,

Phys. Rev. Lett. **79**, 4047 (1997).