Stick-slip waves between elastic and rigid half-spaces

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Abstract

This paper is devoted to the construction of analytic solution of stick-slip waves crossing the interface between an elastic half-space and a rigid one under unilateral contact and Coulomb friction. The method of solution is based on the analytic continuation method of Radok's complex potentials within the framework of steady elastodynamical problems. The governing equations combined with the boundary conditions are reduced to a Riemann-Hilbert problem with discontinuous coefficient and closed-form expressions of the solution are derived. It is found that the existence of solutions depends on the additional velocity V^* which is related to the longitudinal elongation. If V^* is ignored ($V^* = 0$) then there is no solution, if not ($V^* \neq 0$) then it is possible to construct weakly singular solutions satisfying all stick-slip conditions except over a narrow zone where the waves exhibit a crack-like behaviour.

Key words: Stick-slip waves, Unilateral contact, Coulomb friction, Riemann-Hilbert problem, Crack-like behaviour

1. Introduction

This paper is dedicated to Professor L.A. Galin whose scientific contributions are exceptionally rich and widely recognized in contact and friction mechanics [1–5]. Within this field, we present here some of our results and

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open questions related to the problem of stick-slip waves between an elastic half plane and a rigid one [6].

During the two last decades, an important issue emerges in the area of elastodynamic problems involving frictional contact following Coulomb's law, namely the friction-induced vibrations under the form of stick-slip waves. Such cyclic responses, generally accompanied with noise emittence, are closely related to the dynamic bifurcation from the steady response to a cyclic vibration in the sense of Poincar-Hopf bifurcation [7]. In literature, first studies of stick-slip focused on discrete systems, typically the Van-der Pol oscillator. For such mechanical systems, the formation of stick-slip motion is attributed to a static coefficient of friction higher than a kinematic one or to the decay of the kinematic coefficient with the sliding velocity.

Recent investigations of the steady sliding contact between dissimilar elastic half-spaces or between a semi-infinite elastic solid and a rigid one showed that the steady state is dynamically unstable for a *constant* friction coefficient [8–11]. Adams [11] suggested that this dynamic instability is related to the destabilization of the so-called interfacial slip waves. Such waves exist in frictionless contact and are called also the "generalized Rayleigh waves" because they propagate with the Rayleigh wave celerity. Later Ranjith et al [12] demonstrated that, for material combinations where the generalized Rayleigh wave exists, the steady sliding with Coulomb friction is dynamically unstable for an arbitrarily small values of the friction coefficient.

Although the mechanism of flutter instability of friction is now well understood [11, 12], only few works are devoted to the construction of analytic solution of self-excited vibrations for *continuum* media. Without being exhaustive, we note that first solutions of stick-slip or stick-slip-separation waves travelling along the interface between two identical elastic half-spaces, were proposed by Comninou [13, 14]. Later, Freund [15] pointed out a similarity feature between the Comninou's waves and the propagation of interface cracks and showed that the encountered singularities imply energy sources and sinks. Hence, Comninou's solutions were controversial and not physical. For these same reasons, Adams [16] didn't succeed to construct an interfacial stick-slip waves between two dissimilar half-spaces, however, in [17] a family of slip pulse propagating with the celerity of generalized Rayleigh waves is successfully built. Adda-Bedia et al [18] studied the existence of a steady-state slip pulse on finite size between two dissimilar materials. It has been found found that solutions showing a crack-like behavior exist with an arbitrary slip velocity and an arbitrary slip length. For bounded solids, to our

knowledge, the only semi-analytic stick-slip and stick-slip-separation waves are derived in [19–21] in view of the study of two-dimensional brake-like system.

2. Formulation of the problem

Consider an elastic solid, with shear modulus G and waves velocities c_1, c_2 occupying the lower half space Ω^- and sliding against the upper rigid half-space Ω^+ which moves to the right with velocity V, as shown in Fig.1, in plane strain conditions. Unilateral contact and Coulomb friction with constant friction coefficient μ are assumed. It is emphasized that the *interfacial* or *local* coefficient of friction μ is the ratio of shear to normal contact pressure at the interface which would cause local slipping to occur [17]. The elastic body is subjected to remote constant stresses $\tau^*_{yy} < 0$, $\tau^*_{xy} > 0$ such that $\tau^*_{xy} = -\mu^* \tau^*_{yy}$, with $\mu^* < \mu$.

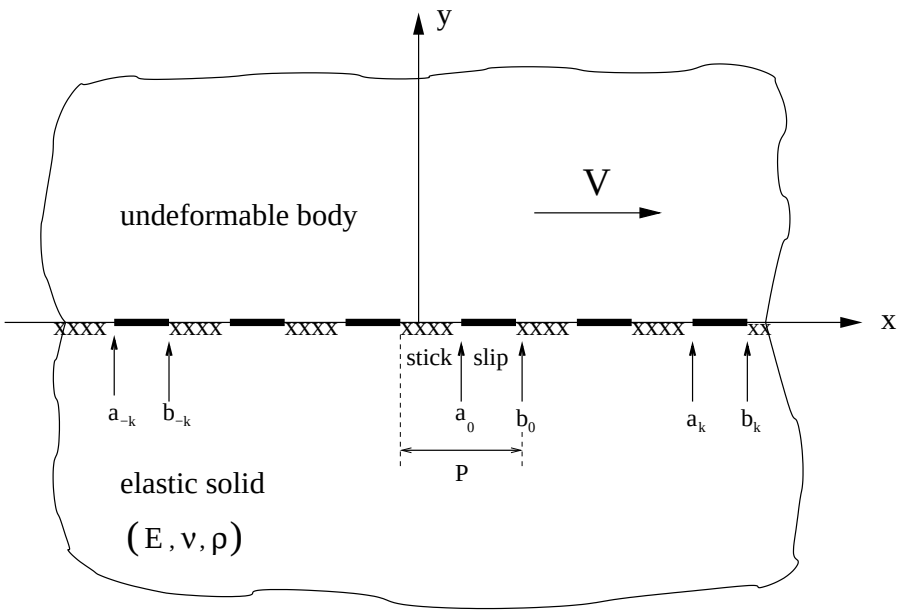


Figure 1: A rigid body sliding on an elastic half-space

Suppose that the interface is entirely a stick zone, then the velocity of points of the interface $y = 0^-$, is $\dot{U}_x = V$. Since the shear stress grows indefinitely, the interface will no longer adhere to the upper non-deformable solid.

On the other hand, suppose that the interface $y = 0^-$ is entirely a sliding one, with the constant static stress ($\tau^*_{xy} > 0$, $\tau^*_{yy} < 0$). Thus the elastic body is at rest under static stress, $\dot{U}_x = 0$. The slip condition of the Coulomb's friction law is not satisfied. Therefore both conditions on the velocity and Coulomb's friction law are incompatible. Thus, we investigate here the possibility of relative motion of the two bodies, due to the existence of periodic stick and slip regions which propagate along the interface with some wave speed c. In this case the quantity μ^* may be interpreted as the apparent coefficient of friction, since sliding occurs with that ratio of applied shear to normal traction [17, 18, 22].

We assume that the periodic stick-slip wave consists in a stick region plus a slip one indefinitely repeated as shown in Fig.1.
Let us denote by SL the set of the slip segments

$$SL = \cdots [a_{-k}, b_{-k}] \bigcup \cdots [a_0, b_0] \bigcup \cdots [a_k, b_k] \bigcup \cdots , \quad k \in \mathbb{N}$$

The remainder part of contact interface is the stick zone and will be refereed by ST. The infinite axis $y = 0$ is oriented in the direction of increasing x and the following notation will be frequently used: $SL^+ = lim_{y\to 0^+} SL$ and $SL^- = lim_{y\to 0^-} SL$. In the same way, one defines ST^+ and ST^-.

If we fix the origin of coordinate frame at $b_{-1} = 0$, we can simply denote any particular slip zone $[a_i, b_i]$ by its generic segment $[a \equiv a_0, b \equiv b_0]$ and any stick zone $[b_{k-1}, a_k]$ by its generic segment $[b_{-1} = 0, a \equiv a_0]$. We note $P = [0, a_0] \cup [a_0, b_0]$.

Together with a fixed frame coordinates (X, Y), we shall use also moving coordinates with the algebraic velocity c. Later we shall se that $c < 0$. Capital letters U_i , Σ_{ij} are for coordinates in fixed frame, small letters for coordinates in moving frame. Hence $X = x + ct$ and $Y = y$. Material derivative in steady state case ($\frac{\partial g}{\partial t} = 0$) is denoted by a dot, $\frac{dg}{dt} \equiv \dot{g} = -c\ g_{,x}$.

In the present work, it is reasonable to assume that the periodic stick-slip regime occurs under the condition discussed above $\dot{U}_x = 0$ which is satisfied, not locally in the whole interface, but only in the homogenized sense. That is the "mean" value of the velocity over a stick-slip period $[0, b]$ vanishes

$$\left\langle \dot{U}_x \right\rangle = 0 \tag{1}$$

where $\langle f \rangle = \frac{1}{|P|} \int_P f dx$.

This vanishing mean value ensures that the stress field of the lower elastic body does not blow up in time.
In the steady state case, Eq(1) is equivalent to $\langle -c\ U_{x,x} \rangle = 0$ or to the periodicity condition

$$U(b) = U(0) \tag{2}$$

Condition $\dot{U}_x = V$ is only satisfied in the stick zones ST, while the sliding condition $\Sigma_{xy}[U] = -\mu \Sigma_{yy}[U]$ is only satisfied in the slip zones SL. The incompatibility between boundary conditions is relaxed through homogenization so that the effective or homogenized "friction" coefficient μ^* can be smaller that the true one.

Let the material velocity in the elastic body at interface be decomposed into the sum

$$\dot{U} = V^* + \dot{u} = V^* - c\ u_{x,x} \tag{3}$$

where V^* is some velocity defined hereafter. We shall show that the presence of longitudinal elongation ε^*_{xx} enables to solve the periodic stick-slip problem. If this elongation is ignored by setting $\varepsilon^*_{xx} = 0$ in the equation, we will show that there is no solution. From elastic law, the elongation in the x-direction is

$$\varepsilon^*_{xx} = u^*_{x,x} = -\frac{\nu(1+\nu)}{E}\ \tau^*_{yy} > 0 \tag{4}$$

Notice that the mode deformation expressed by (4) is derived from Hooke's law with constant stress $\tau^*_{xy} = -\mu\ \tau^*_{yy}$ and $\tau^*_{xx} = 0$. In the present work the most important stresses are $(\tau^*_{yy}, \tau^*_{xy})$ since they are involved in the friction law. The analysis and the generality of the results are not affected and remain valid if one takes $\tau^*_{xx} \neq 0$.

The velocity V^* is defined hereafter by

$$V^* = -c\ u^*_{x,x} = c\ \frac{\nu(1+\nu)}{E}\ \tau^*_{yy} \tag{5}$$

Note that V^* is positive because $c < 0$ as it will be shown hereafter.
Eq(5) establishes a relationship between τ^*_{yy}, c and V^*. Hence, the principal unknowns are the wave celerity c and b for prescribed stresses and V^*.

Let σ_{xy}, σ_{yy} and σ_{xx} be the additional stresses correspondent to the perturbed stick-slip motion. The solution satisfies the following conditions

- Active contact along the interface $y = 0$:

$$u_y = 0 \tag{6}$$

- Material velocity:

$$\dot{U} \equiv -c\ u_{x,x} + V^* = V \quad \text{in the stick region } ST \tag{7}$$

- Sliding condition:

$$\Sigma_{xy}[U] \equiv \sigma_{xy} + \tau^*_{xy} = -\mu(\sigma_{yy} + \tau^*_{yy}) \equiv -\mu\Sigma_{yy}[U] \quad \text{in the slip region } SL \tag{8}$$

- The following inequality:

$$|\Sigma_{xy}[U]| < \mu\ |\Sigma_{yy}[U]| \quad \text{in the stick zone } ST \tag{9}$$

- σ_{xy} and σ_{yy} behaves like

$$\sigma_{ij} = O(1/(x^2 + y^2)) \quad \text{at infinity} \tag{10}$$

We search c, ∇u, σ_{xy}, σ_{yy} periodic satisfying relations (6-10). An addition condition comes from the periodicity of fields. Note that even if the following fields $\dot{U}$, U (displacement), ∇u, σ are periodic, the field u_x, is not. A counter-example can be seen in Fig.2 for the periodic field (continuous line) $u_{x,x} = 1$ in $[0, 1]$, $u_{x,x} = -1$ in $[1, b \neq 0]$ and the non-periodic continuous field u_x (doted line).

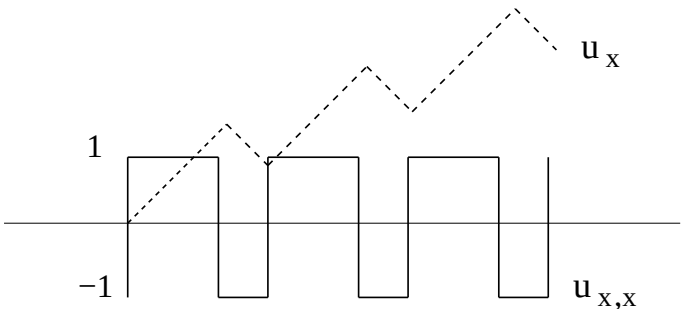


Figure 2: The gradient field $u_{x,x}$ is periodic (continuous line) however the displacement u_x is non-periodic (doted line)

From Eq(3) we get

$$-\ c\ U_{x,x} = -c\ u_{x,x} + V^* \tag{11}$$

Assuming the periodicity condition for the total displacement $U(x)$

$$U_x(b) - U_x(0) = 0 \tag{12}$$

we get $-c\,\big(U_x(b) - U_x(0)\big) = -c(u_x(b) - u_x(0)) + V^* b = 0$ and thus

$$u_x(b) - u_x(0) = \frac{bV^*}{c} \tag{13}$$

Thus, $u_x(x,0)$ is not periodic. The constant $-V^*/c$ has been introduced as the applied elongation ε^*_{xx}. Another interpretation of (13) is the sliding displacement over a period. Remark that Eq.(1) shows that the "mean perturbed velocity" $\langle \dot{u} \rangle$ over a period of length b is opposite to V^*. Thus we recover (13)

$$\langle \dot{u} \rangle = -\frac{c}{b}\Big(u_x(b) - u_x(0)\Big) = -V^* \tag{14}$$

Another interpretation of Eq(1) is to impose the overall "mean" strain $U_{x,x}$ equal to zero, or to require the "mean" strain $\langle u_{x,x} \rangle$ be opposite to the applied strain $u^*_{x,x} = -\dfrac{V^*}{c}$.

Let us now recall the **steady** elastodynamic equations within the framework of homogenous and isotropic elasticity under the plane strain hypothesis. In the moving frame (Oxy) attached to the propagating stick-slip wave $(x = X - ct\,, \quad y = Y)$, one introduces the Radok complex variables

$$z_1 = x + i\beta_1 y \quad ; \quad z_2 = x + i\beta_2 y$$

where $\beta_1 = \sqrt{1 - \dfrac{c^2}{c_1^2}}$, $\beta_2 = \sqrt{1 - \dfrac{c^2}{c_2^2}}$, $i = \sqrt{-1}$ is the imaginary unit number and c_1, c_2 stand for the celerity of longitudinal and shear waves defined respectively by

$$c_1 = \sqrt{\frac{\lambda + 2G}{\rho}}\,, \quad c_2 = \sqrt{\frac{G}{\rho}}$$

G is the shear modulus, λ denotes the Lamé's coefficient and ρ is the mass density.
The displacement and stress fields are given in terms of tow complex poten-

tials ϕ_1 and ϕ_2 as follows [23]

$$u_x = -\frac{1}{G}\,\Re e\Big(\phi_1(z_1) + \frac{1+\beta_2^2}{2}\phi_2(z_2)\Big) \tag{15}$$

$$u_y = \frac{1}{G}\,\Im m\Big(\beta_1\phi_1(z_1) + \frac{1+\beta_2^2}{2\beta_2}\phi_2(z_2)\Big) \tag{16}$$

$$\sigma_{xx} = -2\,\Re e\Big(\frac{2\beta_1^2-\beta_2^2+1}{2}\phi_1'(z_1) + \frac{1+\beta_2^2}{2}\phi_2'(z_2)\Big) \tag{17}$$

$$\sigma_{yy} = (1+\beta_2^2)\,\Re e\Big(\phi_1'(z_1) + \phi_2'(z_2)\Big) \tag{18}$$

$$\sigma_{xy} = 2\,\Im m\Big(\beta_1\phi_1'(z_1) + \frac{(1+\beta_2^2)^2}{4\beta_2}\phi_2'(z_2)\Big) \tag{19}$$

where $\Re e(Z)$ and $\Im m(Z)$ represent respectively the real and the imaginary part of the complex number Z.
The present study focus on the subsonic waves i.e. $c < c_2$. Hence, $\beta_1 > 0$ and $\beta_2 > 0$.

3. Construction of the solution

3.1. Basic equations

Following Bui et al [24, 25], the construction method of the solution is based on the *displacement continuation*. From the condition $u_y = 0$ along the contact interface (real axis where $z_1 = z_2 = z = x \pm i0$) and equation (16) one obtains

$$\beta_1\ \phi_1(z) + \frac{(1+\beta_2^2)}{2\beta_2}\ \phi_2(z) = 0 \tag{20}$$

This equation suggests the following definition for the function ϕ_2 in the physical domain by

$$\phi_2(z_2) := -\frac{2\beta_1\beta_2}{(1+\beta_2^2)}\ \phi_1(z_2) \tag{21}$$

Substitution of (21) in (15-19) results in the following equations on the interface ($y = 0$) where $z_1 = z_2 = z$

$$u_x = -\frac{(1-\beta_1\beta_2)}{G}\,\Re e\Big(\phi_1(z)\Big) \tag{22}$$

$$\sigma_{yy} = (1+\beta_2^2-2\beta_1\beta_2)\,\Re e\Big(\phi_1'(z)\Big) \tag{23}$$

$$\sigma_{xx} = \Big((-1+\beta_2^2)+2\beta_1(\beta_2-\beta_1)\Big)\,\Re e\Big(\phi_1'(z)\Big) \tag{24}$$

$$\sigma_{xy} = \beta_1(1-\beta_2^2)\,\Im m\Big(\phi_1'(z)\Big) \tag{25}$$

Hence, all mechanical fields are determined through the knowledge of the function ϕ_1 and its derivative ϕ_1'.

In the sequel, unless stated otherwise, the notation $\Phi(z) := \phi_1'(z)$ will be used.

We shall search for a complex solution under the form of a Cauchy integral with a distribution $f(t)$

$$\Phi(z) = \frac{1}{2i\pi}\int_a^b \frac{f(t)}{t-z}dt \tag{26}$$

where $z = x + iy$.

It is useful to underline that if $f(t)$ is real then the conjugate function of Φ is given, for z on the real axis, by

$$\overline{\Phi}(z) = -\frac{1}{2i\pi}\int_a^b \frac{f(t)}{t-z}dt = -\Phi(z) \tag{27}$$

Note that (27) may remains valid for some complex functions $f(t)$. It will be the case for the solution in the slip zone as it will be shown hereafter.

3.2. Solution in the slip zone

The sliding state expressed by equation (8) may be written as

$$\sigma_{xy} + \tau^*_{xy} = -\mu(\sigma_{yy} + \tau^*_{yy}) \Rightarrow \sigma_{xy} + \mu\sigma_{yy} = -\tau^*_{xy} - \mu\tau^*_{yy} := T^* \tag{28}$$

The stress T^* can be written as $T^* = \mu^*\ \tau^*_{yy} - \mu\ \tau^*_{yy} = (\mu^* - \mu)\tau^*_{yy} > 0$. It is positive because $\mu^* < \mu$ and $\tau^*_{yy} < 0$.

By setting $\gamma_1 = \beta_1(1-\beta_2^2) > 0$ and $\gamma_2 = 1+\beta_2^2 - 2\beta_1\beta_2$ and by virtue of $\overline{\Phi}^+(z) = -\Phi^+(z)$, Eq.(28) becomes

$$-g\Phi^-(z) + \Phi^+(z) = \frac{2iT^*}{\gamma_1 - i\mu\gamma_2} := f_2(t) \tag{29}$$

where

$$f_2(t) = \frac{2iT^*}{\gamma_1 - i\mu\gamma_2} \tag{30}$$

is a constant and

$$g = -\frac{\gamma_1 + i\mu\gamma_2}{\gamma_1 - i\mu\gamma_2} \tag{31}$$

It is usefull to notice that the coefficient g can be expressed in a different manner

$$g = -e^{2i\pi\alpha} = e^{2i\pi(\alpha+\frac{1}{2})} \ , \quad |g| = 1 \tag{32}$$

with

$$\tan(\alpha\pi) = \mu \ \frac{\gamma_2}{\gamma_1} \tag{33}$$

Equation (29) is called the Riemann-Hilbert problem and has a closed form solution [26]. The appropriate **P-periodic** solution of the homogenous Riemann-Hilbert problem has the following form

$$X(z) = \prod_{k=-\infty}^{+\infty} \Big(z-(a+kb)\Big)^{\frac{1}{2}-\alpha} \Big(z-(k+1)b\Big)^{-\frac{1}{2}+\alpha} \tag{34}$$

involving product of holomorphic functions with a cut along the slip segments SL. Observe that $X(z)$ is regular at $a_k = a+kb$ and weakly singular at $b_k = (k+1)b$. [1]

It easily seen that

$$X(z) = lim_{N\to+\infty} \underbrace{\sum_{k=-N}^{+N} \Big(z-(a+kb)\Big)^{\frac{1}{2}-\alpha} \Big(z-(k+1)b\Big)^{-\frac{1}{2}+\alpha}}_{X^N(z)} = lim_{N\to+\infty} X^N(z) \tag{35}$$

[1] Regular functions at a_k and b_k are not considered for the construction of the solution because the infinite product $X(z)$ is divergent at infinity and thus the proposed approach does not work.

It can be proved for a fixed z, that the sum e of the remaining terms is of order $e = O(1/N)$. Thus $X^N(z)$ tends absolutely to a limit, denoted by $X(z)$ when $N \to \infty$.

Following Mushkhelishvili [26], the general solution of (29) has the form

$$\Phi^{slip}(z) = \frac{f_2}{2i\pi} X(z) \Big\{ \int_{SL^+} \frac{1}{X(t^+)} \frac{dt}{t-z} + C_2 \Big\} \tag{36}$$

where C_2 is an arbitrary constant. Besides, it is easy to prove that (36) satisfies $\overline{\Phi}(z) = -\Phi(z)$.

3.3. Solution in the stick zone

Equating the horizontal velocity in stick zone to $V - V^*$, $u_{x,x} = \dfrac{\partial u_x}{\partial x} = -\dfrac{(V - V^*)}{c}$, and knowing that $\dfrac{\partial(\cdot)}{\partial x} = \dfrac{\partial(\cdot)}{\partial z} + \dfrac{\partial(\cdot)}{\partial \overline{z}}$ we obtain

$$u_{x,x} = -\frac{(1-\beta_1\beta_2)}{2G} \Big\{ \Phi(z) + \overline{\Phi}(\overline{z}) \Big\} = -\frac{(1-\beta_1\beta_2)}{2G} \Big\{ \Phi^-(z) - \Phi^+(z) \Big\} = -\frac{(V - V^*)}{c} \tag{37}$$

The Plemelj formulas permit one to obtain the density function $f_1(t)$

$$f_1(t) = \Phi^+(t) - \Phi^-(t) = -\frac{2G(V - V^*)}{c(1-\beta_1\beta_2)} \tag{38}$$

Note that $f_1(t)$ is a real constant. One possible solution is given by

$$\Phi(z) = \frac{f_1}{2i\pi} \int_{ST} \frac{dt}{t-z} \tag{39}$$

With this form of solution, the displacement at infinity should be logarithmic, and not of order $O(1/z)$ as required and thus does not respect the continuity condition on $y = 0$. Thus the expression of Φ given by (39) must be modified. An adequate choice of a P-periodic, holomorphic and multivalued function is given by

$$X(z) = \prod_{k=-\infty}^{+\infty} \Big(z - (a + kb) \Big)^{\frac{1}{2}-\alpha} \Big(z - (k+1)b \Big)^{-\frac{1}{2}+\alpha}$$

where α is a real number introduced in the previous section by the equation (33). Notice that this function is continuous through the stick segments,

satisfies $X^+(z) = X^-(z)$ for every $z \in ST$ and is constant at infinity (equal to 1).
It then follows

$$\left\{\frac{\Phi^+(z)}{X^+(z)} - \frac{\Phi^-(z)}{X^-(z)}\right\} = -\frac{2G(V-V^*)}{c(1-\beta_1\beta_2)}\frac{1}{X^+(z)} \tag{40}$$

Therefore, the expression of complex potential in the stick zone is given by [26]

$$\Phi^{stick}(z) = \frac{f_1}{2i\pi}X(z)\left\{\int_{ST^+}\frac{dt}{X^+(t)(t-z)} + C_1\right\} \tag{41}$$

where C_1 is constant.

3.4. Stick-slip solution

The stick-slip potential is obtained by sum the two complex solutions Φ^{stick} and Φ^{slip}. We obtain the following Cauchy integral with discontinuous density

$$\Phi(z) = \frac{1}{2i\pi}X(z)\left\{f_1\int_{ST^+}\frac{dt}{X^+(t)(t-z)} + f_2\int_{SL^+}\frac{dt}{X^+(t)(t-z)} + C_0\right\} \tag{42}$$

where C_0 is an arbitrary constant, f_1 and f_2 are constant given by (30) and (38) respectively. Moreover, since the stress field vanishes at infinity then $C_0 = 0$.

Explicit evaluation of $\Phi(z)$ requires the knowledge of the explicit expressions of the Cauchy integrals in (42). One obtains

$$\Phi(z) = f_1\frac{g}{g-1}iX(z)e^{-i\pi\alpha}\left(\frac{1}{Z(z)} - 1\right) + \frac{f_2}{1-g}\left(1 - X(z)\right) \tag{43}$$

where $Z(z)$ is defined by the same infinite product (34) but arranged differently, with cuts along stick zones ST (instead of cuts along slip zones for $X(z)$)

$$Z(z) = \cdots(z-b_{-(k-1)})^{-\beta}(z-a_{-k})^{\beta}\cdots\underbrace{(z-b_{(k-1)})^{-\beta}(z-a_k)^{\beta}}_{Z_k(z)}(z-b_{k)})^{-\beta}\cdots \tag{44}$$

where $\beta = \frac{1}{2} - \alpha$.

Explicit calculation gives the following analytic expression of the stick-slip potential splitted in its real and imaginary parts

$$\begin{aligned}\Phi(z \in SL) &= \Big\{ -\frac{f_1}{2}\left(1-|X(z)|\right) + \frac{T^*}{\gamma_1}\, cos(\pi\alpha)\, |X(z)| \Big\} + \\ & i\Big\{ -\frac{f_1}{2}\,\mu\,\frac{\gamma_2}{\gamma_1}\,(|X(z)|-1) + \frac{T^*}{\gamma_1}\left(1-|X(z)|sin(\pi\alpha)\right)\Big\} \end{aligned} \quad (45)$$

and

$$\Phi(z \in ST) = -\frac{f_1}{2} - i\left\{ -\frac{f_1}{2}\mu\frac{\gamma_2}{\gamma_1} + \frac{f_1}{2\cos(\pi\alpha)}|X(z)| + \frac{T^*}{\gamma_1}\big(1-|X(z)|\big)\right\} \quad (46)$$

It is worth noting that these complex functions (45) and (46) are weakly singular (i.e. square integrable) which avoids the problem of unbounded energy sources or sinks [15] at switching boundaries between stick and slip segments.
Let us remark that the nature of the mathematical singularity was explored in [24, 27] where it was shown that the singularity $(-\frac{1}{2}+\alpha)$ is associated to push-in test of rigid fibre in its elastic matrix material and the singularity $(\frac{1}{2}-\alpha)$ corresponds to pull-out kinematics.

4. Results and discussion

Let us recall that the stick-slip function Φ must behave as $O(1/z^2)$ at infinity then terms of order $O(1/z)$ must be canceled. This condition will be first imposed for the truncated product $X^N(z)$.

For $|z| \to \infty$, after some cumbersome calculus, one obtains

$$\Phi_N^{stick}(z) + \Phi_N^{slip}(z) = -\frac{\beta}{z}\frac{(2N+1)}{g-1}\big\{ f_1 g i e^{-i\pi\alpha}(a-b_{-1}) + f_2(b-a)\big\} \quad (47)$$

Observe that this equation is independent of N and thus it is sufficient to cancel terms of order $O(1/z)$ over a single and an isolated generic stick-slip segment $[b_{-1}=0, a] \cup [a, b]$. This results in

$$f_1 g i e^{-i\pi\alpha}(a-b_{-1}) + f_2(b-a) = 0 \quad (48)$$

or in an equivalent manner

$$\frac{2G(V-V^*)}{c(1-\beta_1\beta_2)}(\gamma_1 + i\mu\gamma_2)e^{-i\pi\alpha}(a-b_{-1}) + 2T^*(b-a) = 0 \quad (49)$$

Dividing (49) by a and setting $b_{-1} = 0$ (i.e. the stick segment starts at the origin O of the coordinate frame) one obtains

$$f_1(\gamma_1 + i\mu\gamma_2)e^{-i\pi\alpha} - 2T^*(\delta - 1) = 0 \tag{50}$$

where $\delta = b/a$. Equation (50) is apparently a complex one in the form $F_1(\delta, c) + iF_2(\delta, c) = 0$, where F_1 and F_2 are real, so that it should be splitted into two real equations. However, exact calculus shows that $F_2 \equiv f_1(-\gamma_1\ sin(\alpha\pi) + \mu\ \gamma_2\ cos(\alpha\pi) = 0$ because $tan(\alpha\pi) = \mu\dfrac{\gamma_2}{\gamma_1}$. Equation (50) is thus reduced to $F_1(\delta, c) = 0$ which provides the first relation between the wave velocity c and δ

$$\frac{G(V - V^*)}{c(1 - \beta_1\beta_2)}(\gamma_1\ cos(\alpha\pi) + \mu\ \gamma_2\ sin(\alpha\pi)) + T^*(\delta - 1) = 0 \tag{51}$$

The second relation between c and δ, which enables complete determination of these unknowns will be set further by setting the continuity condition obtained by periodicity of the total field namely, $U_x(b) = U_x(0)$, so that according to (2), $u_x(b) - u_x(0) = \dfrac{bV^*}{c}$. This permits one to have the second equation allowing complete determination of c and δ. This "closure" condition results in

$$-\frac{V}{c}\delta + \left\{\frac{(V - V^*)}{c} - (1 - \beta_1\beta_2)\ cos(\pi\alpha)\frac{T^*}{2G\gamma_1}\right\}\pi(\frac{1}{2} - \alpha)\frac{(\delta - 1)}{cos(\pi\alpha)} = 0 \tag{52}$$

In conclusion, the unknowns of this stick-slip problem (c, δ) for given loading conditions (T^*, V, V^*) are completely determined by the set of the two equations (51) and (52).

Physical quantities involved in the problem may be scaled as follows

$$a = 1, \quad \delta = \frac{b}{a} \to b > 1, \quad \frac{T^*}{G} \to T, \quad \frac{V}{c_2} \to v, \quad \frac{V^*}{c_2} \to v^*, \quad \frac{c_1}{c_2} \to \overline{c}_1, \quad \frac{c}{c_2} \to \overline{c}$$

We obtain the following final equations

$$-\frac{v}{\overline{c}}b + \left\{\frac{(v - v^*)}{\overline{c}} - (1 - \beta_1\beta_2)\ cos(\pi\alpha)\frac{T}{2\gamma_1}\right\}\pi(\frac{1}{2} - \alpha)\frac{(b - 1)}{cos(\pi\alpha)} = 0 \tag{53}$$

and

$$\frac{(v - v^*)}{\overline{c}(1 - \beta_1\beta_2)}(\gamma_1\ cos(\alpha\pi) + \mu\ \gamma_2\ sin(\alpha\pi)) + T(b - 1) = 0 \tag{54}$$

Following results in literature [9, 10, 20], the stick-slip wave propagates from *rear to front*, i.e. from the right to the left in our problem. Therefore, we shall search for a wave celerity in the range $-c_2 < c < 0$ (or $-1 < \overline{c} < 0$ for the non dimensional equations).

The stick-slip problem $(\overline{c}, b)$, for given loading conditions (T, v, v^*) is solved by the set of the two equations (53) and (54). This problem is highly non linear in the wave velocity c and the uniqueness of the solution cannot be easily proved. Moreover its numerical solution by Newton's method for example, is inaccurate and fails because of singularities. Let us propose a simpler method to solve it. Instead of finding $(\overline{c}, b)$ for a given loading condition (v, v^*, T) (or (v, v^*, τ^*_{xy}) since), we search the solution of the following *inverse problem*:

Find (v, T) *for any given pair* $(\overline{c}, b)$ *and given* v^*.

Therefore, one obtains a linear algebraic system

$$\Big[\pi(\frac{1}{2}-\alpha)\frac{(b-1)}{cos(\pi\alpha)}-b\Big]\frac{v}{\overline{c}}-\Big[\pi(\frac{1}{2}-\alpha)(b-1)\frac{(1-\beta_1\beta_2)}{2\gamma_1}\Big]T=\pi(\frac{1}{2}-\alpha)\frac{(b-1)}{cos(\pi\alpha)}\frac{v^*}{\overline{c}} \tag{55}$$

and

$$\Big[2\frac{(\gamma_1\ cos(\alpha\pi)+\mu\ \gamma_2\ sin(\alpha\pi))}{(1-\beta_1\beta_2)}\Big]\frac{v}{\overline{c}}+[b-1]T=2\frac{(\gamma_1\ cos(\alpha\pi)+\mu\ \gamma_2\ sin(\alpha\pi))}{(1-\beta_1\beta_2)}\frac{v^*}{\overline{c}} \tag{56}$$

The system (55-56) is mathematically well-posed in the sense that we have two equations for two unknowns (v, T). Note that once T is known the stresses τ^*_{yy} and τ^*_{xy} are easily obtained.

The solution is straightforward and depends on the determinant of the linear system

$$D(b,\overline{c})=\frac{(b-1)}{\gamma_1\,\overline{c}}H(b,\overline{c}) \tag{57}$$

where

$$H(b,\overline{c})=-\gamma_1 b+\gamma_1\pi(\frac{1}{2}-\alpha)\frac{(b-1)}{cos(\pi\alpha)}+\pi(\frac{1}{2}-\alpha)\big(\gamma_1\ cos(\alpha\pi)+\mu\ \gamma_2\ sin(\alpha\pi)\big) \tag{58}$$

It is easy to establish that $H(b,\overline{c}) > 0\ \forall b > 1$ and $\forall \overline{c} \in]-1, 0[$. Thus the determinant $D(b,\overline{c})$ is strictly negative $\forall b > 1$ and $\forall \overline{c} \in]-1, 0[$.

We shall distinguish two cases: $v^* = 0$ and $v^* \neq 0$.

4.1. Solution for $v^* = 0$

When $v^* = 0$, equations (55-56) are reduced to a homogenous algebraic system. The later have non trivial solutions in the domain $-1 < \overline{c} < 0$ and $b > 1$ if and only if the determinant $D(b, \overline{c})$ vanishes. However, we have checked that $D(b, \overline{c}) < 0$ in the domain $-1 < \overline{c} < 0$ and $b > 1$. Therefore, only the trivial solution $v = 0$ (statics) and $T = 0$ (the loading at the Coulomb limit) exists.
We conclude that no solution exists in the case $v* = 0$. This result agrees with the conclusion of Caroli (2000) for friction of a viscoelastic medium against a rigid half-space where no solution is found.

4.2. Solution for $v^* \neq 0$

For the case $v^* \neq 0$ the solution is straightforward for v and T

$$v = \frac{v^*}{1 + A(b, \overline{c})} \tag{59}$$

where

$$A(b, \overline{c}) = \frac{4b\gamma_1 cos(\alpha\pi)}{\pi(-1+2\alpha)\Big(\mu\gamma_2 sin(2\pi\alpha) + \gamma_1(-1+2b+cos(2\pi\alpha))\Big)}$$

and

$$T = -\frac{4b\gamma_1(\gamma_1 + \gamma_1 cos(2\alpha\pi) + \mu\gamma_2 sin(2\alpha\pi))}{(b-1)(-1+\beta_1\beta_2)} \times \frac{v^*}{\overline{c}\,\Big(4b\gamma_1 cos(\alpha\pi) + \pi(-1+2\alpha)\big(\mu\gamma_2 sin(\pi\alpha) + \gamma_1(-1+2b+cos(\pi\alpha))\big)\Big)} \tag{60}$$

Note that the solution pair (v, T) exists and is unique for any wave velocity such that $-1 < \overline{c} < 0$, any $b > 1$ and for a given v^*.

As can be noted in Eq.(22) and Eq.(23), the gradient $u_{x,x}$ and the stress σ_{yy} are of opposite signs. Since the slip velocity has a negative singularity at $x = b$ (fig.3), the normal stress shows a positive singularity as seen in Fig.4 where Σ_{yy}/G is plotted. Note also that, in the stick zone at the transition point with slip segments, the shear stress is singular and thus the stick

condition (10) is violated over a very small zone, cf. fig.5. A similar result has been pointed out by Adda-Bedia and Ben Amar (2003) in analyzing the existence of a single steady-state slip pulse between two elastic media by the use of Weertman's dislocation formulation.

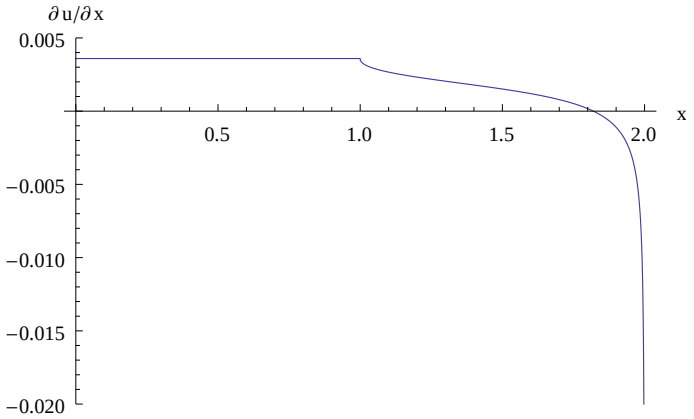


Figure 3: The gradient $u_{x,x}$ over a stick-slip segment: a negative singularity near $b = 2$ with $E = 2.1\,10^{11} Pa$, $\rho = 7850 Kg/m^3$, $\nu = 0.3$, $\mu = 0.5$, $\mu^* = 0.4$, $v^* = 0.001$ and $\overline{c} = -0.7$

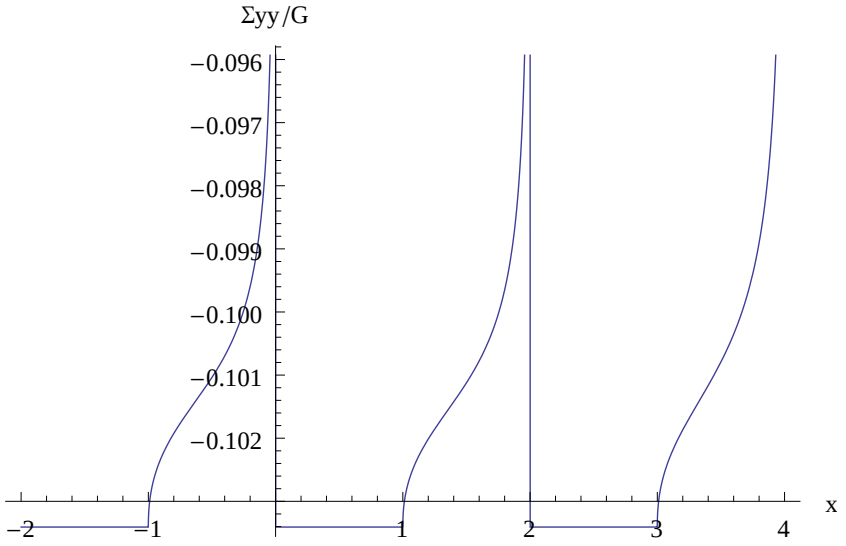


Figure 4: The normalized stress Σ_{yy}/G (for the same numerical data as for Fig.3) over stick-slip segments exhibits a positive singularity at the transition points $SL \to ST$

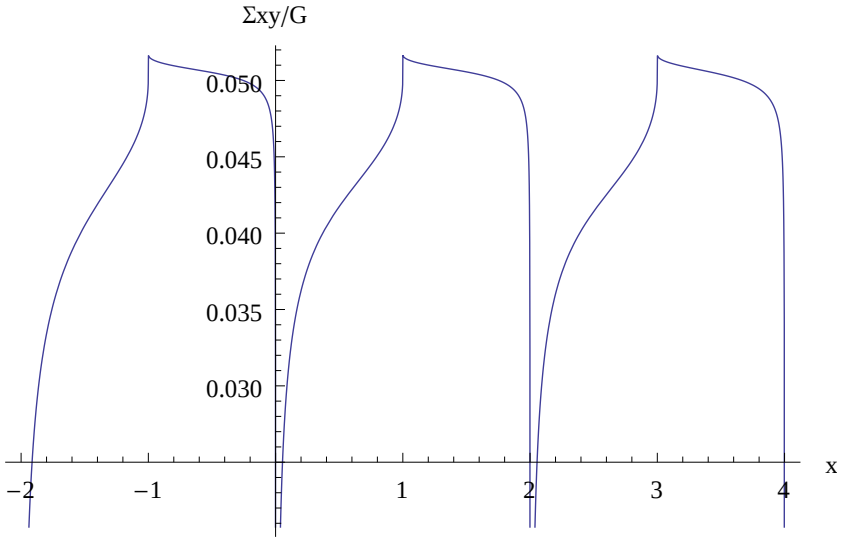


Figure 5: The normalized stress Σ_{xy}/G (for the same numerical data as for Fig.3) over stick-slip segments shows a negative singularity at the transition points $SL \to ST$

It is worth noting that the simplicity of the inverse problem allows an easy parametric study of sensibility of the results under various conditions. We may calculate $R =$ (separation region)$/b$ (where $\Sigma_{yy} > 0$) and S the ratio of the zone length for which (10) is violated by b. They are found very small for a wide range of different combinations of the problem parameters. For example, for fixed $E = 2.1\,10^{11} Pa$, $\rho = 7850 Kg/m^3$, $\nu = 0.3$, $\mu - \mu^* = 0.2$, $v^* = 0.001$ and $\overline{c} = -0.7$, it is found that $R(b = 2) = 6\ 10^{-5}$ and $R(b = 1.05) = 3.1\ 10^{-6}$ while $S(b = 2) = 3\ 10^{-4}$ and $S(b = 1.05) = 6.3\ 10^{-5}$. Besides for fixed $b = 1.5$, $v* = 0.1$, $c = -0.7$ and $\mu = 0.7$ these ratios vary with respect to μ^* as follows: $R(\mu^* = 0.1) = 10^{-3}$ and $R(\mu^* = 0.65) = 3.3\ 10^{-6}$ while $S(\mu^* = 0.1) = 2\ 10^{-2}$ and $S(\mu^* = 0.65) = 6.3\ 10^{-5}$. Finally, the ratios R and S are practically not sensitive to variation of v^* and c.

5. Conclusion

In this paper we have analyzed the stick-slip waves between an elastic half-space and a rigid one under unilateral contact and dry friction. The method of representation of plane strain steady state elastodynamics by Radok's potentials has been used and the stick-slip problem is reduced to a Riemann-Hilbert problem. We have shown that he obtained equation can be solved in a

closed-form under the conditions that a uniform longitudinal strain $\varepsilon^*_{xx} = u^*_{x,x}$ is considered as the consequence of the normal stress τ^*_{yy}. If the strain ε^*_{xx} vanishes or is ignored ($V^* = -cu_{x,x} = 0$) there is no solution. Furthermore, it is shown that the solution is obtained by solving an inverse procedure in which the loading conditions (v, T) is determined as a function of the wave celerity $\overline{c}$, b and v^*. There is no oscillatory behavior of the stress but the solution shows a crack-like behaviour: the shear stress exhibits a negative singularity while the normal stress has a positive singularity at the extremity of the slip region which also means a separation in a narrow zone. This incomplete solution is obtained for "regular-weakly singular" at the ends a_k and b_k respectively.

Contrary to some authors who change the friction law and adopt generally a rate-and-state friction law to overcome this difficulty, our future plans is to conserve the Coulomb friction law and to investigate at least many possible models in the future: (i) Model of stick-slip-separation (ii) Stick-slip-separation-reverse slip waves as already worked out semi-analytically and numerically by Oueslati et al [28] and Nguyen et al [20] (iii) Model of stick-slip with regular functions at a_k and b_k, for which the method outlined in this paper does not work.

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