Fluid Dynamic Limits of the Kinetic Theory of Gases

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Abstract These three lectures introduce the reader to recent progress on the hydrodynamic limits of the kinetic theory of gases. Lecture 1 outlines the main mathematical results in this direction, and explains in particular how the Euler or Navier-Stokes equations for compressible as well as incompressible fluids, can be derived from the Boltzmann equation. It also presents the notion of renormalized solution of the Boltzmann equation, due to P.-L. Lions and R. DiPerna, together with the mathematical methods used in the proofs of the fluid dynamic limits. Lecture 2 gives a detailed account of the derivation by L. Saint-Raymond of the incompressible Euler equations from the BGK model with constant collision frequency [L. Saint-Raymond, Bull. Sci. Math. **126** (2002), 493–506]. Finally, lecture 3 sketches the main steps in the proof of the incompressible Navier-Stokes limit of the Boltzmann equation, connecting the DiPerna-Lions theory of renormalized solutions of the Boltzmann equation with Leray's theory of weak solutions of the Navier-Stokes system, following [F. Golse, L. Saint-Raymond, J. Math. Pures Appl. **91** (2009), 508–552]. As is the case of all mathematical results in continuum mechanics, the fluid dynamic limits of the Boltzmann equation involve some basic properties of isotropic tensor fields that are recalled in Appendices 1-2.

Introduction

The purpose of these lecture notes is to introduce the reader to a series of recent mathematical results on the fluid dynamic limits of the Boltzmann equation.

The idea of looking for rigorous derivations of the partial differential equations of fluid mechanics from the kinetic theory of gases goes back to D. Hilbert. In his 6th problem presented in his plenary address at the 1900 International Congress of

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Mathematicians in Paris [46], he gave this as an example of "axiomatization" of physics. In Hilbert's own words

> [...] Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes [...] which lead from the atomistic view to the laws of motion of continua.

Hilbert himself studied this problem; his contributions include an important theorem (Theorem 3.2 below) on the linearization at uniform equilibrium states of the Boltzmann collision integral, together with a systematic asymptotic expansion method still widely used more than 100 years after his article [47] appeared (see section 1.2).

Of course, after Hilbert's 1900 address [46], physics evolved in such a way that, while the existence of atoms was no longer questioned as in the days of L. Boltzmann and J.C. Maxwell, the classical kinetic theory of gases could no longer be considered as a good example of an "axiom of physics".

In fact, the Boltzmann equation of the kinetic theory of gases can be rigorously derived as an asymptotic limit of Newton's second law of motion written for each molecule in a gas [51]. Certainly Newton's laws of motion can be regarded as an axiom of classical mechanics. However, the idea that the Boltzmann equation could be viewed as a consequence of Newton's laws of motion appeared for the first time in a remarkable paper by H. Grad [41], almost half a century after Hilbert formulated his problems.

But while Hilbert's original question lost some of its interest from the point of view of theoretical physics, it has gained a lot of importance with the various applications of kinetic modeling in modern technology (such as rarefied gas dynamics in the context of space flight, plasma physics, neutron transport in fissile material, semiconductor physics ...) Readers interested in applications of rarefied gas dynamics will find a lot of information in [75].

These lectures are focused on fluid dynamic limits of the kinetic theory of gases that can be formulated in terms of global solutions, and for any initial data within a finite distance to some uniform equilibrium state, measured in terms of relative entropy.

The first lecture describes how the most important partial differential equations of fluid dynamic (such as the Euler, Stokes or Navier-Stokes equations) can be derived as scaling limits of the Boltzmann equation. While this first lecture will review the basic mathematical properties of the Boltzmann equation, it leaves aside all the technicalities involved in either the proof of existence of global solutions of the Boltzmann equation, or the proof of the fluid dynamic limits. This first lecture is concluded with an overview of some of the main mathematical tools and methods used in the proof of these limits.

Lectures 2 gives a rather detailed account of the proof of the incompressible Euler limit of (a model of) the Boltzmann equation, following [66]. Lecture 3 provides a much less detailed account of the derivation of the incompressible Navier-Stokes equation from the Boltzmann equation. This last lecture follows [40] rather closely. Since the Navier-Stokes limit involves a much heavier technical apparatus than the

Euler limit, the presentation of the proof in lecture 3 will be deliberately impressionistic. However, these lecture notes will give precise references to the main results in [40], and can therefore be used as a reader's guide for this last reference. Lectures 2 and 3 make a connection between three different notions of weak solutions of either the Boltzmann, or the Euler, or the Navier-Stokes equations: the Leray solutions of the Navier-Stokes equation, the DiPerna-Lions renormalized solutions of the Boltzmann equation, and the more recent notion of "dissipative solutions" of the Euler equation proposed by P.-L. Lions.

There are several other introductions to the material contained in these notes, including C. Villani's report at the Bourbaki seminar [77], which is less focused on the Euler and Navier-Stokes limits, and gives the main ideas used in the proofs of these limits with less many details as in the present notes. The lecture notes by C.D. Levermore and the author [33] leave aside the material presented in lecture 2 (the incompressible Euler limit), and give a more detailed account of the material presented in lecture 1. The various sets of lecture notes or monographs by L. Saint-Raymond and the author [39, 28, 69] are much more detailed and give a more comprehensive picture of the Boltzmann equation and its various fluid dynamic limits.

1 Lecture 1: Formal Derivations

This first lecture is a slightly expanded version of the author's Harold Grad Lecture [30], with an emphasis on mathematical tools and methods used in the theory of the Boltzmann equation and of its fluid dynamic limits.

For the sake of simplicity, the exposition is limited to the case of a (monatomic) hard sphere gas. More general collision processes, involving radial, binary intermolecular potentials satisfying Grad's angular cutoff assumption [43] can also be considered. The interested reader is referred to the original articles for a more complete account of these results.

1.1 The Boltzmann Equation

In the kinetic theory of gases (proposed by J.C. Maxwell and L. Boltzmann), the state at time t of a monatomic gas is defined by its *distribution function* $F \equiv F(t,x,v) \geq 0$, which is the density (with respect to the Lebesgue measure $dxdv$) of gas molecules with velocity $v \in \mathbf{R}^3$ to be found at the position $x \in \mathbf{R}^3$ at time t. The evolution of the distribution function is governed by the Boltzmann equation.

If the effect of external forces (such as gravity) is negligible, the Boltzmann equation for the distribution function F takes the form

$$\partial_t F + v \cdot \nabla_x F = \mathscr{C}(F),$$

where the right-hand side is known as "the collision integral".

Assuming that all gas molecules are identical and that collisions between gas molecules are elastic, hard sphere binary collisions, the collision integral is defined on functions of the velocity variable v that are rapidly decaying at infinity by the formula

$$\mathscr{C}(f)(v) := \tfrac{d^2}{2} \iint_{\mathbf{R}^3\times\mathbf{S}^2} (f(v')f(v'_*) - f(v)f(v_*))|(v-v_*)\cdot\omega| dv_* d\omega\,,$$

where $d/2$ is the molecular radius, and where

$$\begin{cases} v' \equiv v'(v,v_*,\omega) := v - (v-v_*)\cdot\omega\omega\,, \\ v'_* \equiv v'_*(v,v_*,\omega) := v_* + (v-v_*)\cdot\omega\omega\,. \end{cases} \tag{1}$$

(The notation $d\omega$ designates the uniform measure on the unit sphere $\mathbf{S}^2$.)

This collision integral is extended to distribution functions (depending also on the time and position variables t and x) by the formula

$$\mathscr{C}(F)(t,x,v) := \mathscr{C}(F(t,x,\cdot))(v)\,.$$

The physical meaning of this definition is that, except for the molecular radius appearing in front of the collision integral $\mathscr{C}(F)$, gas molecules are considered as point particles in kinetic theory, so that collisions are purely local and instantaneous. Besides, the collision integral is quadratic in the distribution function, because the Boltzmann equation is valid in a scaling regime where collisions other than binary can be neglected.

With the definition above of $v' \equiv v'(v,v_*,\omega)$ and $v'_* \equiv v'_*(v,v_*,\omega)$, for each $v,v_* \in \mathbf{R}^3$ and $\omega \in \mathbf{S}^2$, one has the following conservation laws, whose physical interpretation is obvious (since all the gas molecules are identical and therefore have the same mass):

$$\begin{aligned} v' + v'_* &= v + v_*\,, && \text{conservation of momentum,} \\ |v'|^2 + |v'_*|^2 &= |v|^2 + |v_*|^2\,, && \text{conservation of energy.} \end{aligned}$$

Definition 1.1 *A collision invariant is a function $\phi : \mathbf{R}^3 \to \mathbf{R}$ satisfying*

$$\phi(v'(v,v_*,\omega)) + \phi(v'_*(v,v_*,\omega)) = \phi(v) + \phi(v_*)\,, \quad \textit{for all } v,v_* \in \mathbf{R}^3,\ \omega \in \mathbf{S}^2\,.$$

Obviously $\phi(v) \equiv 1$, $\phi(v) \equiv v_j$ for $j = 1,2,3$ and $\phi(v) = |v|^2$ are collision invariants (because elastic hard sphere collisions preserve the number of particles, together with the total momentum and energy of each colliding particle pair). A remarkable feature of the Boltzmann equation is that the converse is true (under some regularity assumption on ϕ).

Theorem 1.2 *Let $\phi \in C(\mathbf{R}^3)$; then ϕ is a collision invariant if and only if there exists $a,c \in \mathbf{R}$ and $b \in \mathbf{R}^3$ such that*

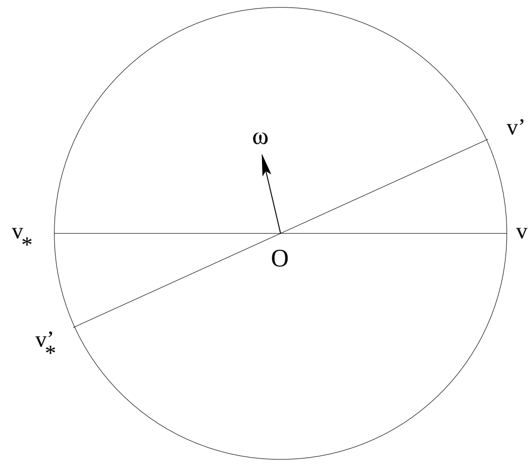

Fig. 1 The velocities v, v_*, v', v'_* in the center of mass reference frame, and the geometrical meaning of the unit vector ω. The relative velocities $v - v_*$ and $v' - v'_*$ are exchanged by the reflection with respect to the plane orthogonal to ω.

$$\phi(v) = a + b \cdot v + c|v|^2 .$$

The proof of this result is rather involved; it is an extension of the well known proof that the only function $\psi \in C(\mathbf{R})$ such that

$$\psi(x+y) = \psi(x) + \psi(y) \text{ for all } x, y \in \mathbf{R}, \quad \text{ and } \psi(1) = 1$$

is the identity, i.e.

$$\psi(x) = x \quad \text{ for each } x \in \mathbf{R}^3 .$$

See for instance [19], chapter II.6, especially pp. 74–77.

Theorem 1.3 *For each measurable $f \equiv f(v)$ rapidly decaying as $|v| \to \infty$ and each collision invariant $\phi \in C(\mathbf{R}^3)$ with at most polynomial growth as $|v| \to \infty$, one has*

$$\int_{\mathbf{R}^3} \mathscr{C}(f)\phi(v)dv = 0 .$$

Proof. Denoting $f = f(v)$, $f' = f(v')$, $f_* = f(v_*)$ and $f'_* = f(v'_*)$, one has

$$\begin{aligned}
\int_{\mathbf{R}^3} \mathscr{C}(f)\phi dv &= \tfrac{d^2}{2} \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} \phi(f' f'_* - f f_*)|(v - v_*) \cdot \omega| dv dv_* d\omega \\
&= \tfrac{d^2}{2} \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} \tfrac{1}{2}(\phi + \phi_*)(f' f'_* - f f_*)|(v - v_*) \cdot \omega| dv dv_* d\omega ,
\end{aligned}$$

since the collision integrand is symmetric in v, v_*.

Since $(v - v_*) \cdot \omega = -(v' - v'_*) \cdot \omega$ and $(v, v_*) \mapsto (v', v'_*)(v, v_*, \omega)$ is a linear isometry of $\mathbf{R}^6$ for each $\omega \in \mathbf{S}^2$ (by the conservation of energy), the Lebesgue measure is invariant under the change of variables $(v,v_*)\mapsto(v',v'_*)(v,v_*,\omega)$, which is an involution. Therefore

$$\begin{aligned}&\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}\tfrac{1}{2}(\phi+\phi_*)(f'f'_*-ff_*)|(v-v_*)\cdot\omega|dvdv_*d\omega\\=&\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}\tfrac{1}{2}(\phi'+\phi'_*)(ff_*-f'f'_*)|(v-v_*)\cdot\omega|dvdv_*d\omega\,,\end{aligned}$$

which implies the

Formula of collision observables

$$\int_{\mathbf{R}^3}\mathscr{C}(f)\phi dv=\tfrac{d^2}{2}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}\tfrac{1}{4}(\phi+\phi_*-\phi'-\phi'_*)(f'f'_*-ff_*)|(v-v_*)\cdot\omega|dvdv_*d\omega\,.$$

The conclusion of Theorem 1.3 follows from the definition of collision invariants.

Specializing the identity in the theorem above to $\phi(v)\equiv 1, v_k$ for $k=1,2,3$ or $\phi(v)=|v|^2$, for each $f\equiv f(v)$ rapidly decaying as $|v|\to\infty$, one has

$$\int_{\mathbf{R}^3}\mathscr{C}(f)dv=\int_{\mathbf{R}^3}\mathscr{C}(f)v_kdv=\int_{\mathbf{R}^3}\mathscr{C}(f)|v|^2dv=0\,,\quad k=1,2,3\,.$$

Thus, solutions F of the Boltzmann equation that are rapidly decaying together with their first order derivatives in t and x as $|v|\to\infty$ satisfy the local conservation laws

$$\left\{\begin{aligned}\partial_t\int_{\mathbf{R}^3}Fdv+\operatorname{div}_x\int_{\mathbf{R}^3}vFdv=0\,,&\quad\text{(mass)}\\\partial_t\int_{\mathbf{R}^3}vFdv+\operatorname{div}_x\int_{\mathbf{R}^3}v\otimes vFdv=0\,,&\quad\text{(momentum)}\\\partial_t\int_{\mathbf{R}^3}\tfrac{1}{2}|v|^2Fdv+\operatorname{div}_x\int_{\mathbf{R}^3}v\tfrac{1}{2}|v|^2Fdv=0\,.&\quad\text{(energy)}\end{aligned}\right.$$

The next most important property of the Boltzmann equation is Boltzmann's H Theorem. This is a rigorous mathematical result bearing on solutions of the Boltzmann equation, which corresponds to the second principle of thermodynamics. The second principle of thermodynamics states that the entropy of an isolated system can only increase until the system reaches an equilibrium state. However there is no general formula for the entropy production. In the context of the kinetic theory of gases, Boltzmann's H Theorem gives an explicit formula for the entropy production in terms of the distribution function.

Theorem 1.4 (Boltzmann's H Theorem) *If $f\equiv f(v)$ is a measurable function on $\mathbf{R}^3$ such that $0<f=O(|v|^{-m})$ for all $m>0$ and $\ln f=O(|v|^n)$ for some $n>0$ as $|v|\to\infty$, then*

$$\int_{\mathbf{R}^3}\mathscr{C}(f)\ln f dv\le 0\,.$$

Moreover

$$\int_{\mathbf{R}^3}\mathscr{C}(f)\ln f dv=0\Leftrightarrow\mathscr{C}(f)=0\Leftrightarrow f\text{ Maxwellian,}$$

i.e. there exists $\rho, \theta > 0$ *and* $u \in \mathbf{R}^3$ *s.t.*

$$f(v) = \mathscr{M}_{(\rho,u,\theta)}(v) := \frac{\rho}{(2\pi\theta)^{3/2}} \exp\left(-\frac{|v-u|^2}{2\theta}\right).$$

Proof. Applying the formula of collision observables with $\phi = \ln f$ shows that

$$\int_{\mathbf{R}^3} \mathscr{C}(f) \ln f dv = \tfrac{d^2}{2} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \tfrac{1}{4}(f'f'_* - ff_*) \ln\left(\frac{ff_*}{f'f'_*}\right) |(v-v_*)\cdot\omega| dv dv_* d\omega .$$

Since $z \mapsto \ln z$ is increasing on $\mathbf{R}_+^*$

$$(f'f'_* - ff_*) \ln\left(\frac{ff_*}{f'f'_*}\right) = (f'f'_* - ff_*)(\ln(ff_*) - \ln(f'f'_*)) \leq 0 ,$$

so that

$$\int_{\mathbf{R}^3} \mathscr{C}(f) \ln f dv \leq 0 .$$

Now for the equality case:

$$\int_{\mathbf{R}^3} \mathscr{C}(f) \ln f dv = 0 \Leftrightarrow f'f'_* = ff_* \Leftrightarrow \ln f \text{ is a collision invariant}$$
$$\Leftrightarrow \mathscr{C}(f) = 0 .$$

If $\ln f$ is a collision invariant and $f \to 0$ as $|v| \to \infty$, then

$$\ln f(v) = a + b\cdot v + c|v|^2 \quad \text{ with } c < 0 ,$$

so that $f(v) = \mathscr{M}_{(\rho,u,\theta)}(v)$ with

$$\theta = -\tfrac{1}{2c} , \quad u = -\tfrac{b}{2c} , \quad \text{ and } \rho = \left(\tfrac{\pi}{|c|}\right)^{3/2} e^{a+|b|^2/4c} .$$

Thus

$$\int_{\mathbf{R}^3} \mathscr{C}(f) \ln f dv \leq 0 \Leftrightarrow f \text{ is a Maxwellian.}$$

In particular, positive solutions F of the Boltzmann equation that are rapidly decaying together with their first order derivatives in t and x as $|v| \to \infty$ and such that $\ln F$ has at most polynomial growth in $|v|$ satisfy the local entropy inequality

$$\partial_t \int_{\mathbf{R}^3} F \ln F dv + \operatorname{div}_x \int_{\mathbf{R}^3} vF \ln F dv = \int_{\mathbf{R}^3} \mathscr{C}(f) \ln f dv \leq 0 .$$

1.2 The Compressible Euler Limit

We shall study solutions of the Boltzmann equation that are slowly varying in both the time and space variables.

In other words, we want to study solutions F of the Boltzmann equation of the form

$$F(t,x,v) = F_\varepsilon(\varepsilon t, \varepsilon x, v)\,,$$

assuming

$$\partial_{\hat{t}} F_\varepsilon\,,\ \nabla_{\hat{x}} F_\varepsilon = O(1)\,, \quad \text{with } (\hat{t},\hat{x}) = (\varepsilon t, \varepsilon x)\,.$$

Since F is a solution of the Boltzmann equation, one has

$$\partial_{\hat{t}} F_\varepsilon + v \cdot \nabla_{\hat{x}} F_\varepsilon = \frac{1}{\varepsilon}\mathscr{C}(F_\varepsilon)\,.$$

Hilbert [47] proposed to seek F_ε as a formal power series in ε with smooth coefficients:

$$F_\varepsilon(\hat{t},\hat{x},v) = \sum_{n\geq 0} \varepsilon^n F_n(\hat{t},\hat{x},v)\,.$$

In the literature on kinetic theory, this expansion bears the name of *Hilbert's expansion*. It is the most systematic method used to investigate all fluid dynamic limits of the Boltzmann equation (see [74, 75]).

The leading order term in Hilbert's expansion is of the form

$$F_0(\hat{t},\hat{x},v) = \mathscr{M}_{(\rho,u,\theta)(\hat{t},\hat{x})}(v)\,,$$

where (ρ,u,θ) is a solution of the compressible Euler system

$$\left\{\begin{aligned} \partial_{\hat{t}}\rho + \operatorname{div}_{\hat{x}}(\rho u) &= 0\,,\\ \rho(\partial_{\hat{t}} u + u\cdot\nabla_{\hat{x}} u) + \nabla_{\hat{x}}(\rho\theta) &= 0\,,\\ \partial_{\hat{t}}\theta + u\cdot\nabla_{\hat{x}}\theta + \tfrac{2}{3}\theta \operatorname{div}_{\hat{x}} u &= 0\,. \end{aligned}\right. \tag{2}$$

The Hilbert series is a formal object — in particular, its radius of convergence in ε may be, and often is 0. A mathematical proof of the compressible Euler method based on some variant of Hilbert's expansion truncated at some finite order in ε was proposed by R. Caflisch [17].

While fairly direct and natural, Caflisch's approach to the compressible Euler limit meets with the following difficulties:

a) the truncated Hilbert expansion may be negative for some $\hat{t},\hat{x},v$;
b) the k-th term in Hilbert's expansion is of order $F_k = O(|\nabla_{\hat{x}}^k F_0|)$;
c) generic solutions of Euler's equations lose regularity in finite time (see [73]).

Statement (a) follows from a close inspection of Caflisch's asymptotic solution at time $t = 0$; statement b) implies that the Hilbert expansion method can be used in the case of smooth solutions of the compressible Euler system, while statements (b-c) suggest that the Hilbert expansion breaks down in finite time for generic smooth solutions of the compressible Euler system.

There is another approach to the compressible Euler limit. T. Nishida studied the Cauchy problem for the scaled Boltzmann equation in [63]:

$$\begin{cases} \partial_{\hat{t}} F_\varepsilon + v \cdot \nabla_{\hat{x}} F_\varepsilon = \frac{1}{\varepsilon} \mathscr{C}(F_\varepsilon) \,, \\ F_\varepsilon(0, \hat{x}, \hat{v}) = \mathscr{M}_{(\rho^{in}, u^{in}, \theta^{in})(\hat{x})}(v) \,, \end{cases} \tag{3}$$

for analytic $(\rho^{in}, u^{in}, \theta^{in})$. Nishida's idea is to apply the Nirenberg-Ovsyannikov [61, 62] abstract variant of the Cauchy-Kovalevska theorem.

He proved that the Cauchy problem (3) has a unique solution on a time interval $[0, T^*]$ with $T^* > 0$ independent of ε, and that

$$F_\varepsilon(\hat{t}, \hat{x}, v) \to \mathscr{M}_{(\rho, u, \theta)(\hat{t}, \hat{x})}(v)$$

as $\varepsilon \to 0$, where (ρ, u, θ) is the solution of the compressible Euler system with initial data $(\rho^{in}, u^{in}, \theta^{in})$.

It is interesting to compare the Hilbert expansion method and the Caflisch proof with Nishida's.

Caflisch's method leads to a family F_ε of solutions of the scaled Boltzmann equation that converges to a Maxwellian whose parameters satisfy the compressible Euler system on the same time interval as that on which the Euler solution remains smooth.

However, these solutions fail to be everywhere nonnegative; besides the choice of the initial condition $F_\varepsilon\big|_{\hat{t}=0}$ is seriously constrained to "well prepared data". This difficulty was later alleviated by M. Lachowicz [50].

In Nishida's method, we can choose $F_\varepsilon\big|_{\hat{t}=0}$ to be any local Maxwellian with analytic parameters, and F_ε remains everywhere nonnegative.

However the uniform existence time T^* can be a priori smaller than the time during which the Euler solution remains smooth. Besides, analytic regularity is physically unsatisfying.

The works of Caflisch and Nishida obviously raise the question of what happens to the family of solutions of the Boltzmann equation in the vanishing ε limit after the onset of shock waves in the solution of the Euler system. For instance the Cauchy problem for the Euler equations of gas dynamics is known to have global solutions defined for all initial data with small enough total variation, in space dimension 1. These solutions are constructed by Glimm's method [27, 58].

Of course, weak solutions of a hyperbolic system of conservation laws such as the Euler equations of gas dynamics may fail to be uniquely determined by their initial data. For instance, weak solutions can include unphysical shock waves. In the case of gas dynamics, the notion of entropy provides precisely the criteron used to eliminate the possibility of unphysical shock waves. The following elementary observation shows that, under rather weak assumptions, weak solutions of the Euler equations of gas dynamics originating from solutions of the Boltzmann equation satisfy the entropy criterion.

Theorem 1.5 (C. Bardos - F. Golse [5]) *Let $\rho^{in} \geq 0$, $\theta^{in} > 0$ (resp. u^{in}) be measurable functions (resp. vector fields) defined a.e. on $\mathbf{R}^3$ such that*

$$\int_{\mathbf{R}^3} (1+|u^{in}|)(|u^{in}|^2 + \theta^{in} + |\ln \rho^{in}| + |\ln \theta^{in}|) d\hat{x} < \infty .$$

For each $\varepsilon > 0$, let F_ε be a solution of the Cauchy problem (3) satisfying the local conservation laws of mass momentum and energy. Assume that

$$F_\varepsilon \to F \text{ a.e. on } \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3 ,$$

and that

$$\int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} F_\varepsilon(\hat{t},\hat{x},v) dv \to \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} F(\hat{t},\hat{x},v) dv$$

in the sense of distributions on $\mathbf{R}^3$, uniformly on $[0,T]$ for each $T > 0$, while

$$\int_{\mathbf{R}^3} \begin{pmatrix} v \otimes v \\ v|v|^2 \end{pmatrix} F_\varepsilon dv \to \int_{\mathbf{R}^3} \begin{pmatrix} v \otimes v \\ v|v|^2 \end{pmatrix} F dv$$

and

$$\int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \end{pmatrix} F_\varepsilon \ln F_\varepsilon dv \to \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \end{pmatrix} F \ln F dv$$

in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3$. Then

- *the limit F is of the form*
$$F = \mathscr{M}_{(\rho,u,\theta)}$$
where (ρ,u,θ) is a weak solution of the system of Euler equations of gas dynamics (2) (with perfect gas equation of state), with initial data
$$(\rho,u,\theta)\big|_{t=0} = (\rho^{in}, u^{in}, \theta^{in}) ,$$
- *the solution (ρ,u,θ) of the system of Euler equations so obtained satisfies the entropy condition*
$$\partial_{\hat{t}} \left(\rho \ln \left(\frac{\rho}{\theta^{3/2}} \right) \right) + \operatorname{div}_{\hat{x}} \left(\rho u \ln \left(\frac{\rho}{\theta^{3/2}} \right) \right) \leq 0 .$$

The key observation in this result is that

$$\begin{aligned} 0 \geq \partial_{\hat{t}} \int_{\mathbf{R}^3} F_\varepsilon \ln F_\varepsilon dv + \operatorname{div}_{\hat{x}} \int_{\mathbf{R}^3} v F_\varepsilon \ln F_\varepsilon dv \\ \to \partial_{\hat{t}} \int_{\mathbf{R}^3} F \ln F dv + \operatorname{div}_{\hat{x}} \int_{\mathbf{R}^3} v F \ln F dv \end{aligned}$$

in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3$ as $\varepsilon \to 0$, while

$$\int_{\mathbf{R}^3} \mathscr{M}_{(\rho,u,\theta)} \ln \mathscr{M}_{(\rho,u,\theta)} dv = \rho \ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right) - \tfrac{3}{2}\rho\,,$$
$$\int_{\mathbf{R}^3} v\mathscr{M}_{(\rho,u,\theta)} \ln \mathscr{M}_{(\rho,u,\theta)} dv = \rho u \ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right) - \tfrac{3}{2}\rho u\,.$$

(In other words, Boltzmann's H function specialized to Maxwellian distribution functions coincides with the entropy density for a perfect monatomic gas).

Of course, the assumption that $F_\varepsilon \to F$ a.e. is extremely strong, and verifying it remains a major open problem. However, the purpose of this theorem is not the convergence itself to some solution of the Euler equations, but the fact that all solutions of the Euler equations obtained in this way satisfy the entropy condition.

In addition to the system of Euler's equations of gas dynamics, several other fluid dynamic equations can be derived from the Boltzmann equation. We shall review these derivations in the next sections.

1.3 From Boltzmann to Compressible Navier-Stokes

First we seek to derive viscous corrections to the Euler system from the Boltzmann equation. In order to do so, we use the Chapman-Enskog expansion — a variant of Hilbert's. (See [42] and especially chapter V.3 in [19].) This asymptotic expansion in powers of ε takes the form

$$F_\varepsilon(\hat{t},\hat{x},v) \simeq \sum_{n=0}^{N} \varepsilon^n \Phi_n[\mathbf{P}_\varepsilon^N(\hat{t},\hat{x})](v) =: F_\varepsilon^N(\hat{t},\hat{x},v)\,,$$

where

$$\int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \Phi_n[\mathbf{P}](v)dv = \begin{cases} \mathbf{P} & \text{if } n=0\,, \\ 0 & \text{if } n>0\,, \end{cases} \tag{4}$$

and

$$\partial_{\hat{t}} F_\varepsilon^N + v \cdot \nabla_{\hat{x}} F_\varepsilon^N = \frac{1}{\varepsilon}\mathscr{C}(F_\varepsilon^N) + O(\varepsilon^N)\,. \tag{5}$$

At variance with Hilbert's expansion, the coefficients of the successive powers of ε in the Chapman-Enskog expansion depend on ε (except for the 0th order term, which is the local Maxwellian with parameters governed by the compressible Euler system, and therefore coincides with the 0th order term in the Hilbert expansion). These coefficients are completeley determined by their moments of order ≤ 2 in the velocity variable (4) and by the fact that F_ε^N is an asymptotic solution of the Boltzmann equation (5) to within an order $O(\varepsilon^N)$ (in the formal sense).

In particular, for $N = 2$, one finds that

$$F_\varepsilon(\hat{t},\hat{x},v) \simeq \mathscr{M}_{(\rho_\varepsilon,u_\varepsilon,\theta_\varepsilon)} - \varepsilon\mathscr{M}_{(1,u_\varepsilon,\theta_\varepsilon)}\alpha(|V_\varepsilon|,\theta_\varepsilon)A(V_\varepsilon)\cdot\nabla_x u_\varepsilon$$
$$- 2\varepsilon\mathscr{M}_{(1,u_\varepsilon,\theta_\varepsilon)}\beta(|V_\varepsilon|,\theta_\varepsilon)B(V_\varepsilon)\cdot\nabla_x\sqrt{\theta_\varepsilon}$$
$$+ O(\varepsilon^2)\,,$$

where

$$V_\varepsilon := \frac{V-u_\varepsilon}{\sqrt{\theta_\varepsilon}}\,, \quad A(z) = z^{\otimes 2} - \tfrac{1}{3}|z|^2\,, \quad B(z) = \tfrac{1}{2}(|z|^2-5)z\,.$$

The functions $\alpha(\theta,r)$ and $\beta(\theta,r)$ are obtained by solving two integral equations involving the Boltzmann collision integral linearized about the Maxwellian state $\mathscr{M}_{(1,u,\theta)}$. We refer to Appendix 2 for more details on this matter.

The compressible Navier-Stokes equations take the form

$$\left\{\begin{aligned} &\partial_{\hat{t}}\rho_\varepsilon + \operatorname{div}_{\hat{x}}(\rho_\varepsilon u_\varepsilon) = 0\,,\\ &\partial_{\hat{t}}(\rho_\varepsilon u_\varepsilon) + \operatorname{div}_{\hat{x}}(\rho_\varepsilon u_\varepsilon^{\otimes 2}) + \nabla_{\hat{x}}(\rho_\varepsilon\theta_\varepsilon)\\ &\qquad = \varepsilon\operatorname{div}(\mu(\theta_\varepsilon)D(u_\varepsilon))\,,\\ &\partial_{\hat{t}}(\rho_\varepsilon(\tfrac{1}{2}|u_\varepsilon|^2 + \tfrac{3}{2}\theta_\varepsilon)) + \operatorname{div}_{\hat{x}}(\rho_\varepsilon u_\varepsilon(\tfrac{1}{2}|u_\varepsilon|^2 + \tfrac{5}{2}\theta_\varepsilon))\\ &\qquad = \varepsilon\operatorname{div}_{\hat{x}}(\kappa(\theta_\varepsilon)\nabla_{\hat{x}}\theta_\varepsilon) + \varepsilon\operatorname{div}_{\hat{x}}(\mu(\theta_\varepsilon)D(u_\varepsilon)u_\varepsilon)\,, \end{aligned}\right.$$

where

$$D(u) = \nabla_{\hat{x}}u + (\nabla_{\hat{x}}u)^T - \tfrac{2}{3}\operatorname{div}_{\hat{x}}u\,I\,.$$

These equations are obtained from the local conservation laws of mass, momentum and energy for the Chapman-Enskog expansion of F_ε truncated at order 2.

Notice that the viscosity and heat diffusion terms are $O(\varepsilon)$ in this scaling. In other words, compressible Navier-Stokes equations are *not a limit* of the Boltzmann equation, but a correction of the compressible Euler at the first order in ε.

The formulas giving the viscosity and heat diffusion coefficients are worth a few comments. They are

$$\begin{aligned} \mu(\theta) &= \tfrac{2}{15}\theta\int_0^\infty \alpha(\theta,r)r^6 e^{-r^2/2}\tfrac{dr}{\sqrt{2\pi}}\,,\\ \kappa(\theta) &= \tfrac{1}{6}\theta\int_0^\infty \beta(\theta,r)r^4(r^2-5)^2 e^{-r^2/2}\tfrac{dr}{\sqrt{2\pi}}\,. \end{aligned} \tag{6}$$

In the hard sphere case (which is the only case considered in these lectures), one finds

$$\mu(\theta) = \mu(1)\sqrt{\theta}\,, \qquad \kappa(\theta) = \kappa(1)\sqrt{\theta}\,. \tag{7}$$

(See Appendix 2 for the details.)

1.4 Global Existence Theory for the Boltzmann Equation

All the hydrodynamic limits that we consider below bear on the Boltzmann equation posed in the whole Euclidean space $\mathbf{R}^3$. Specifically, we are concerned with solutions of the Boltzmann equation which converge to some uniform Maxwellian equilibrium as $|x| \to \infty$. Without loss of generality, by Galilean invariance of the Boltzmann equation and with an appropriate choice of units of time and length, one can assume that this Maxwellian equilibrium is $\mathscr{M}_{(1,0,1)}$.

For simplicity, we shall henceforth use the notation

$$M := \mathscr{M}_{(1,0,1)} .$$

There are various ways of imposing the condition on the solution of the Boltzmann equation as $|x| \to \infty$. In the sequel, we retain the weakest possible notion of convergence to equilibrium at infinity. Perhaps the best reason for this choice is that this notion of "convergence to equilibrium at infinity" is conveniently expressed in terms of Boltzmann's H Theorem.

Specifically, we consider the notion of *relative entropy* (of the distribution function F with respect to the Maxwellian equilibrium M):

$$H(F|M) := \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \left[F \ln\left(\frac{F}{M}\right) - F + M \right] dx dv$$

Notice that the integrand is a nonnegative measurable function defined a.e. on $\mathbf{R}^3 \times \mathbf{R}^3$, so that $H(F|M)$ is a well defined element of $[0,\infty]$ for each nonnegative measurable function F defined a.e. on $\mathbf{R}^3 \times \mathbf{R}^3$.

We are interested in the Cauchy problem

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = \mathscr{C}(F), \quad (t,x,v) \in \mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3 , \\ F(t,x,v) \to M \quad \text{as } |x| \to +\infty , \\ F\big|_{t=0} = F^{in} . \end{cases}$$

The convergence of the distribution function F to the Maxwellian equilibrium M as $|x| \to \infty$ is replaced with the condition

$$H(F|M)(t) < +\infty$$

for all $t \geq 0$. Because of Boltzmann's H theorem and the local conservation laws of mass momentum and energy, rapidly decaying solutions F of the the Boltzmann equation satisfy

$$H(F|M)(t) \leq H(F|M)(0) .$$

In other words, our substitute for the convergence of the distribution function to the uniform Maxwellian equilibrium M as $|x| \to \infty$ is stable under the time evolution of the Boltzmann equation.

R. DiPerna and P.-L. Lions [25, 54] made the following important observation: for each $r > 0$, one has

$$\iint_{|x|+|v|\le r} \frac{\mathscr{C}(F)}{\sqrt{1+F}} dvdx \le \mathrm{C} \iint_{|x|\le r} (-\mathscr{C}(F)\ln F + (1+|v|^2)F)dxdv\,.$$

This suggests considering the following (very weak) notion of solution of the Boltzmann equation.

Definition 1.6 (Renormalized solutions of the Boltzmann equation) *A renormalized solution relative to M of the Boltzmann equation is a nonnegative function $F \in C(\mathbf{R}_+, L^1_{loc}(\mathbf{R}^3 \times \mathbf{R}^3))$ satisfying $H(F(t)|M) < +\infty$ and*

$$M(\partial_t + v \cdot \nabla_x)\Gamma(F/M) = \Gamma'(F/M)\mathscr{C}(F)$$

in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3$, for each $\Gamma \in C^1(\mathbf{R}_+)$ s.t.

$$\Gamma'(Z) \le \frac{C}{\sqrt{1+Z}}\,.$$

The main advantage of this notion of solution is the following global existence theorem, which holds for any initial distribution function with finite relative entropy with respect to the Maxwellian equilibrium M. The following theorem summarizes several results by R. DiPerna-P.-L. Lions [25], P.-L. Lions [54] and P.-L. Lions-N. Masmoudi [56].

Theorem 1.7 (R. DiPerna-P.-L. Lions-N. Masmoudi) *For each measurable initial data $F^{in} \ge 0$ a.e. such that $H(F^{in}|M) < +\infty$, there exists a renormalized solution relative to M of the Boltzmann equation with initial data F^{in}. It satisfies*

$$\begin{cases} \partial_t \int_{\mathbf{R}^3} F dv + \operatorname{div}_x \int_{\mathbf{R}^3} vF dv = 0\,, \\ \partial_t \int_{\mathbf{R}^3} vF dv + \operatorname{div}_x \int_{\mathbf{R}^3} v \otimes vF dv + \operatorname{div}_x m = 0\,, \end{cases}$$

where $m = m^T \ge 0$ is a matrix-valued Radon measure on $\mathbf{R}_+ \times \mathbf{R}^3$, and the entropy inequality

$$H(F(t)|M) + \int_{\mathbf{R}^3} \operatorname{trace} m(t) - \int_0^t \iint_{\mathbf{R}^3\times\mathbf{R}^3} \mathscr{C}(F)\ln F dsdxdv \le H(F^{in}|M)\,.$$

With this notion of solution of the Boltzmann equation, we shall establish the various hydrodynamic limits of the Boltzmann equation where the distribution function is in a weakly nonlinear regime about some uniform Maxwellian equilibrium.

1.5 The Acoustic Limit

The first result on the acoustic limit of the Boltzmann equation in the regime of renormalized solutions can be found in [10]. This early result, valid only in the case of bounded collision kernels, was shortly thereafter extended to more general collision kernels including all hard potentials satisfying Grad's cutoff assumption [43], and in particular the hard sphere case.

Theorem 1.8 (F. Golse-C.D. Levermore [32]) *Let F_ε be a family of renormalized solutions of the Cauchy problem for the Boltzmann equation with initial data*

$$F_\varepsilon\big|_{t=0} = \mathscr{M}_{(1+\delta_\varepsilon\rho^{in}(\varepsilon x),\delta_\varepsilon u^{in}(\varepsilon x),1+\delta_\varepsilon\theta^{in}(\varepsilon x))}$$

for $\rho^{in}, u^{in}, \theta^{in} \in L^2(\mathbf{R}^3)$ and $\delta_\varepsilon|\ln\delta_\varepsilon|^{1/2} = o(\sqrt{\varepsilon})$. When $\varepsilon \to 0$,

$$\frac{1}{\delta_\varepsilon}\int_{\mathbf{R}^3}\left(F_\varepsilon\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},v\right)-M\right)(1,v,\tfrac{1}{3}|v|^2-1)dv \to (\rho,u,\theta)(t,x)$$

in $L^1_{loc}(\mathbf{R}_+\times\mathbf{R}^3)$ for all $t\geq 0$, where $\rho,u,\theta \in C(\mathbf{R}_+;L^2(\mathbf{R}^3)$ satisfy the acoustic system

$$\begin{cases} \partial_t\rho + \mathrm{div}_x u = 0\,, & \rho\big|_{t=0} = \rho^{in}\,,\\ \partial_t u + \nabla_x(\rho+\theta) = 0\,, & u\big|_{t=0} = u^{in}\,,\\ \frac{3}{2}\partial_t\theta + \mathrm{div}_x u = 0\,, & \theta\big|_{t=0} = \theta^{in}\,. \end{cases}$$

1.6 The Incompressible Euler Limit

Steady solutions (ρ,u,θ) of the acoustic system are obviously triples $(\rho,u,\theta) \equiv (\rho(x),u(x),\theta(x))$ satisfying the conditions

$$\mathrm{div}\,u = 0\,, \quad \text{and } \nabla(\rho+\theta) = 0\,.$$

The second constraint implies that $\rho+\theta$ =Const.. In fact, with the additional assumption that $\rho,\theta \in L^2(\mathbf{R}^3)$, one has

$$\rho+\theta = 0\,.$$

This observation suggests that, if the fluctuations around the equilibrium $(1,0,1)$ of density, velocity field and temperature satisfy the conditions above, the acoustic and vortical modes in the moments of the distribution function should decouple in the long time limit, and lead to some incompressible flow.

Of course, this does not mean that the gas is incompressible, but only that its motion is the same as that of an incompressible fluid with constant density. This observation is made rigorous by the following theorem.

Theorem 1.9 (L. Saint-Raymond [68]) *Let $u^{in} \in H^3(\mathbf{R}^3)$ s.t. $\operatorname{div} u^{in} = 0$ and let $u \in C([0,T];H^3(\mathbf{R}^3))$ satisfy*

$$\left\{\begin{aligned} &\partial_t u + u\cdot\nabla_x u + \nabla_x p = 0\,, \quad \operatorname{div}_x u = 0\,,\\ &u\big|_{t=0} = u^{in}\,. \end{aligned}\right.$$

Let F_ε be a family of renormalized solutions of the Cauchy problem for the Boltzmann equation with initial data

$$F_\varepsilon\big|_{t=0} = \mathscr{M}_{(1,\delta_\varepsilon u^{in}(\varepsilon x),1)}$$

for $\delta_\varepsilon = \varepsilon^\alpha$ with $0<\alpha<1$. Then, in the limit as $\varepsilon\to 0$, one has

$$\frac{1}{\delta_\varepsilon}\int_{\mathbf{R}^3} vF_\varepsilon\left(\frac{t}{\varepsilon\delta_\varepsilon},\frac{x}{\varepsilon},v\right)dv \to u(t,x) \text{ in } L^\infty([0,T];L^1_{loc}(\mathbf{R}^3))\,.$$

1.7 The (Time-Dependent) Stokes Limit

The previous limit neglects viscous dissipation in the gas. Viscous dissipation and heat diffusion are observed on a longer time scale. We first treat the case where the nonlinearity is weak even after taking the fluid dynamic limit. This limit is described by the following theorem. Observe that the time scale in this result is $1/\varepsilon^2$, which is large compared to the time scale $1/\varepsilon\delta_\varepsilon$ used in the incompressible Euler limit. On the other hand, the size δ_ε of the fluctuations is $o(\varepsilon)$, i.e. much smaller than in the case of the incompressible Euler limit, where it is $\gg \varepsilon$. Thus the nonlinearity is so weak in this case that it vanishes in the fluid dynamic limit.

Theorem 1.10 (F. Golse-C.D. Levermore [32]) *Let F_ε be a family of renormalized solutions of the Cauchy problem for the Boltzmann equation with initial data*

$$F_\varepsilon\big|_{t=0} = \mathscr{M}_{(1-\delta_\varepsilon\theta^{in}(\varepsilon x),\delta_\varepsilon u^{in}(\varepsilon x),1+\delta_\varepsilon\theta^{in}(\varepsilon x))}\,,$$

where $\delta_\varepsilon|\ln\delta_\varepsilon| = o(\varepsilon)$ and $(u^{in},\theta^{in}) \in L^2\times L^\infty(\mathbf{R}^3)$ s.t. $\operatorname{div}_x u^{in} = 0$. Then, in the limit as $\varepsilon\to 0$, one has

$$\frac{1}{\delta_\varepsilon}\int_{\mathbf{R}^3}\left(F_\varepsilon\left(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon},v\right)-M\right)(v,\tfrac{1}{3}|v|^2-1)dv \to (u,\theta)(t,x) \text{ in } L^1_{loc}\,,$$

where

$$\left\{\begin{aligned} &\partial_t u + \nabla_x p = \nu\Delta_x u\,, \quad \operatorname{div}_x u = 0\,, && u\big|_{t=0} = u^{in}\,,\\ &\tfrac{5}{2}\partial_t\theta = \kappa\Delta_x\theta\,, && \theta\big|_{t=0} = \theta^{in}\,. \end{aligned}\right.$$

The viscosity and heat conductivity are given by the formulas

$$\nu = \tfrac{1}{5}\mathscr{D}^*(v\otimes v - \tfrac{1}{3}|v|^2 I)\,, \qquad \kappa = \tfrac{2}{3}\mathscr{D}^*(\tfrac{1}{2}(|v|^2-5)v)\,, \tag{8}$$

where $\mathscr{D}$ is the Dirichlet form of the linearized collision operator

$$\mathscr{D}(\Phi) = \tfrac{1}{8}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} |\Phi+\Phi_* - \Phi' - \Phi'_*|^2 |(v-v_*)\cdot\omega| MM_* dv dv_* d\omega\,,$$

and $\mathscr{D}^*$ is its Legendre dual.

It should be noticed that P.-L. Lions and N. Masmoudi [56] had independently obtained a version of the above theorem with the motion equation only, i.e. without deriving the heat equation for θ.

1.8 Incompressible Navier-Stokes Limit

Finally, we discuss the case where viscous dissipation and heat diffusion are observed in the fluid dynamic limit, together with the nonlinear convection term. This follows from a scaling assumption where the length and time scale are respectively $1/\varepsilon$ and $1/\varepsilon^2$ (corresponding to the invariance scaling for the heat equation), while the size of the fluctuation is precisely of order ε. Thus the asymptotic regime under consideration is *weakly nonlinear at the level of the kinetic theory of gases, but fully nonlinear at the level of fluid dynamics*. These scaling assumptions correspond exactly to the invariance scaling for the incompressible Navier-Stokes motion equation.

Theorem 1.11 (F. Golse-L. Saint-Raymond [38, 40]) *Let F_ε be a family of renormalized solutions of the Cauchy problem for the Boltzmann equation with initial data*

$$F_\varepsilon\big|_{t=0} = \mathscr{M}_{(1-\varepsilon\theta^{in}(\varepsilon x),\varepsilon u^{in}(\varepsilon x),1+\varepsilon\theta^{in}(\varepsilon x))}\,,$$

where $(u^{in},\theta^{in}) \in L^2\times L^\infty(\mathbf{R}^3)$ s.t. $\mathrm{div}_x u^{in} = 0$. For some subsequence $\varepsilon_n \to 0$, one has

$$\frac{1}{\varepsilon_n}\int_{\mathbf{R}^3}\left(F_{\varepsilon_n}\left(\frac{t}{\varepsilon_n^2},\frac{x}{\varepsilon_n},v\right) - M\right)(v,\tfrac{1}{3}|v|^2-1)dv \to (u,\theta)(t,x)$$

weakly in $L^1_{loc}(\mathbf{R}_+\times\mathbf{R}^3)$, where (u,θ) is a "Leray solution" with initial data (u^{in},θ^{in}) of

$$\begin{cases} \partial_t u + \mathrm{div}_x(u\otimes u) + \nabla_x p = \nu\Delta_x u\,, \quad \mathrm{div}_x u = 0\,, \\ \\ \tfrac{5}{2}(\partial_t\theta + \mathrm{div}_x(u\theta)) = \kappa\Delta_x\theta\,. \end{cases}$$

The viscosity ν in heat diffusion κ in this theorem are given by the same formulas (8) as in the case of the time dependent Stokes limit.

We recall the notion of "Leray solution" of the Navier-Stokes-Fourier system. A Leray solution of the Navier-Stokes-Fourier system above is a couple (u,θ) of elements of $C(\mathbf{R}_+;w-L^2(\mathbf{R}^3))\cap L^2(\mathbf{R}_+;w-H^1(\mathbf{R}^3))$ that is a solution in the sense of distributions and satisfies the Leray inequality below:

Leray inequality

$$\frac{1}{2}\int_{\mathbf{R}^3}(|u|^2+\tfrac{5}{2}|\theta|^2)(t,x)dx+\int_0^t\int_{\mathbf{R}^3}(\nu|\nabla_x u|^2+\kappa|\nabla_x\theta|^2)dxds$$
$$\leq\frac{1}{2}\int_{\mathbf{R}^3}(|u^{in}|^2+\tfrac{5}{2}|\theta^{in}|^2)(t,x)dx\,.$$

This notion of "Leray solution" of the Navier-Stokes-Fourier system finds its origin in the pioneering work of J. Leray [52] on the incompressible Navier-Stokes equations. These solutions bear considerable resemblance with

The reader should be aware that the terminology of "incompressible Navier-Stokes limit" is misleading from the physical viewpoint. It is true that the motion equation satisfied by the velocity field u coincides with the Navier-Stokes equation for an incompressible fluid with constant density. However, the diffusion coefficient in the temperature equation is $3/5$ of its value for an incompressible fluid with the same heat capacity and heat conductivity. The difference comes from the work of the pressure: see the detailed discussion of this subtle point in [30] on pp. 22–23, and especially in [74] (footnote 6 on p. 93) and [75] (footnote 43 on p. 107, together with section 3.7.2). However, the system obtained in the limit has the same mathematical structure than the Navier-Stokes-Fourier system for incompressible fluids, and we shall therefore abuse the terminology of incompressible limit in that case — although it is improper from the strict physical point of view.

The derivation of the acoustic, incompressible Euler, Stokes and Navier-Stokes equations from global (renormalized) solutions of the Boltzmann equation is a program started by Bardos-Golse-Levermore [9].

As for the incompressible Navier-Stokes limit, partial results were obtained by Bardos-Golse-Levermore [7, 8, 9], P.-L. Lions-N. Masmoudi [56] before the complete proof by F. Golse-L. Saint-Raymond appeared in [38, 40]. Subsequently, the validity of this limit was extended to the case of weak cutoff potentials (hard and soft), by C.D. Levermore-N. Masmoudi [53].

In the regime of smooth solutions, the incompressible Navier-Stokes limit for small initial data (a case where Leray solutions are known to be smooth globally in time) had been obtained by C. Bardos-S. Ukai [13]. In the same regime, short time convergence was obtained by A. DeMasi-R. Esposito-J. Lebowitz [23] by an argument similar to Caflisch's for the compressible limit, i.e. by means of a truncated Hilbert expansion.

The various scalings on the Boltzmann equation and the corresponding fluid dynamic limits are summarized in the table below. In all the scaling limits presented above, the small parameter ε is the ratio of the molecular mean free path to some characteristic, macroscopic length scale in the flow, known as the Knudsen number and denoted Kn. The parameter δ_ε entering the initial condition, as in $\mathscr{M}_{(1,\delta_\varepsilon u^{in},1)}$ measures the scale of fluctuations of the velocity field in terms of the velocity scale defined by the background temperature 1, i.e. the speed of sound $\sqrt{\frac{5}{3}}$. Therefore δ_ε

can be regarded as the Mach number (denoted Ma) associated to the initial state of the gas. Finally, the fluid dynamic limits described above may involve a different scaling of the time and space variables. Whenever one considers the distribution function F scaled as $F(t/\varepsilon\lambda_\varepsilon, x/\varepsilon, v)$, the additional scaling parameter λ_ε acting on the time variable can be viewed as the Strouhal number (denoted Sh), following the terminology introduced by Y. Sone [75].

The ratio of viscous dissipation to the strength of nonlinear advection in a fluid is measured by a dimensionless parameter called the Reynolds number, denoted Re. Specifically, $\mathrm{Re} = UL/\nu$, where U and L are respectively the typical velocity and length scales in the fluid flow, while ν is the kinematic viscosity of the fluid. The Reynolds, Mach and Knudsen numbers are related by the following relation:

Von Karman relation

$$\mathrm{Kn} = a\frac{\mathrm{Ma}}{\mathrm{Re}}$$

where a is some "absolute number" (such as $\sqrt{\pi}\dots$)

This important observation explains why the compressible Navier-Stokes equation cannot be obtained as a hydrodynamic *limit* of the Boltzmann equation, but just as a first order correction of the compressible Euler limit. Indeed, the hydrodynamic limit assumes that $\mathrm{Kn} \to 0$; if one seeks a regime where the viscosity coefficient remains positive uniformly as $\mathrm{Kn} \to 0$, then $\mathrm{Re} = O(1)$. This implies that $\mathrm{Ma} \to 0$, so that the limiting velocity field is necessarily divergence-free. In other words, one can only obtain in this way the incompressible Navier-Stokes equations, and not the compressible Navier-Stokes system.

The fluid dynamic regimes presented above are sumarized in the following table.

Table 1 The various incompressible fluid dynamic regimes of the Boltzmann equation in terms of the dimensionless parameters Kn (Knudsen number), Ma (Mach number), Re (Reynolds number) and Sh (Strouhal number).

Boltzmann equation $\mathrm{Kn} = \varepsilon \ll 1$		
von Karman relation $\mathrm{Ma}/\mathrm{Kn} = \mathrm{Re}$		
Ma	Sh	Hydrodynamic limit
$\delta_\varepsilon \ll 1$	1	Acoustic system
$\delta_\varepsilon \ll \varepsilon$	ε	Stokes system
$\delta_\varepsilon \gg \varepsilon$	δ_ε	Incompressible Euler equations
ε	ε	Incompressible Navier-Stokes equations

In the next two lectures, we shall discuss in more detail the incompressible Euler and the incompressible Navier-Stokes-Fourier limits.

1.9 Mathematical Tools: an Overview

We conclude this first lecture with a quick overview of the mathematial notions and methods used in the proof of these limits.

1.9.1 Local Conservation Laws

At the formal level, an important step in deriving fluid dynamic models from the Boltzmann equation is to start from the local conservation laws implied by the Boltzmann equation, which are recalled below for the reader's convenience:

$$\partial_t \int_{\mathbf{R}^3} F_\varepsilon \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} dv + \operatorname{div}_x \int_{\mathbf{R}^3} F_\varepsilon \begin{pmatrix} v \\ v \otimes v \\ v\frac{1}{2}|v|^2 \end{pmatrix} dv = 0 .$$

For instance, if one knows that

$$F_\varepsilon \to F \text{ a.e. pointwise}$$

as $\varepsilon \to 0^+$, Boltzmann's H Theorem implies that

$$\int_0^\infty \iint \mathscr{C}(F) \ln F \, dx dv dt = 0 ,$$

and thus

$$F \equiv \mathscr{M}_{(\rho,u,\theta)(t,x)}(v) .$$

This implies the following "closure relations": in other words, one expresses

$$\int_{\mathbf{R}^3} F \begin{pmatrix} v \otimes v \\ v\frac{1}{2}|v|^2 \end{pmatrix} dv \quad \text{in terms of } \int_{\mathbf{R}^3} F_\varepsilon \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} dv .$$

Because the renormalization procedure is a purely local change of unkown function, it destroys the delicate, nonlocal symmetries in the Boltzmann collision integral. For this reason, it is yet unknown at the time of this writing whether renormalized solutions of the Boltzmann equation satisfy all the local conservation laws above. They are only known to satisfy the local conservation of mass

$$\partial_t \int_{\mathbf{R}^3} F + \operatorname{div}_x \int_{\mathbf{R}^3} vF \, dv = 0 .$$

Instead of the usual local conservation laws of momentum and energy, renormalized solutions of the Boltzmann equation satisfy

$$\partial_t \int_{\mathbf{R}^3} \Gamma\left(\frac{F_\varepsilon}{M}\right)\begin{pmatrix} v \\ \frac{1}{2}|v|^2 \end{pmatrix} M dv + \operatorname{div}_x \int_{\mathbf{R}^3} \Gamma\left(\frac{F_\varepsilon}{M}\right)\begin{pmatrix} v\otimes v \\ \frac{1}{2}|v|^2 v \end{pmatrix} M dv$$
$$= \int_{\mathbf{R}^3} \Gamma'\left(\frac{F_\varepsilon}{M}\right)\mathscr{C}(F_\varepsilon)\begin{pmatrix} v \\ \frac{1}{2}|v|^2 \end{pmatrix} dv\,.$$

An important step in the proof of all the hydrodynamic limits described above will be a) to prove that the r.h.s. of the equalities above vanishes as $\varepsilon \to 0$ and b) that one recovers the usual conservation laws of momentum and energy *in the hydrodynamic limit*, i.e. as $\varepsilon \to 0$.

1.10 Strong Compactness Tools

Since the Navier-Stokes equations are nonlinear, strong compactness (in the Lebesgue L^1_{loc} space) of number density fluctuations is needed in order to pass to the limit in nonlinearities.

The tool for obtaining this compactness is the method of velocity averaging (V. Agoshkov [1], F. Golse-B. Perthame-R. Sentis [35], F. Golse-P.-L. Lions-B. Perthame-R. Sentis [34]), adapted to the L^1 setting. The main statement needed for our purposes is essentially the theorem below.

Theorem 1.12 (F. Golse-L. Saint-Raymond [37]) *Assume that $f_n \equiv f_n(x,v)$ and $v\cdot\nabla_x f_n$ are bounded in $L^1(\mathbf{R}^N_x \times \mathbf{R}^N_v)$, while f_n is bounded in $L^1(\mathbf{R}^N_x; L^p(\mathbf{R}^N_v))$ for some $p > 1$. Then*

a) f_n is weakly relatively compact in $L^1_{loc}(\mathbf{R}^N_x \times \mathbf{R}^N_v)$; and

b) for each $\phi \in C_c(\mathbf{R}^N)$, the sequence of velocity averages

$$\int_{\mathbf{R}^N} f_n(x,v)\phi(v)dv$$

is strongly relatively compact in $L^1_{loc}(\mathbf{R}^N)$.

Observe that the velocity averaging theorem above only gives the strong compactness in L^1_{loc} of *moments* of the sequence of distribution functions f_n, and not of distribution functions themselves.

However, the bound on the entropy production coming from Boltzmann's H Theorem shows that the fluctuations of number densities approach the manifold of infinitesimal Maxwellians (i.e. the tangent linear space of the manifold of Maxwellian equilibrium distribution functions at $M := \mathscr{M}_{(1,0,1)}$). Infinitesimal Maxwellians are — exactly like Maxwellian distribution functions — parametrized by their moments of order ≤ 2 in the v variables, and this explains why strong compactness of the

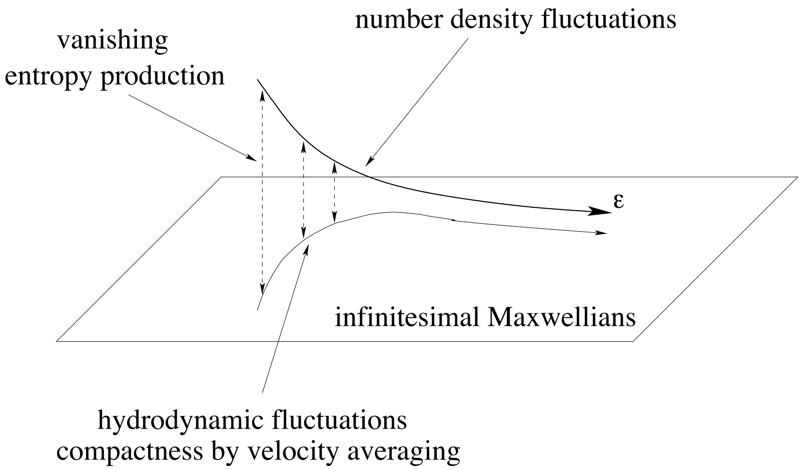

Fig. 2 The family of number density fluctuations approaching the linear manifold of infinitesimal Maxwellian equilibria.

moments of the fluctuations of number density about the uniform Maxwellian equilibrium M is enough for the Navier-Stokes limit.

This will be discussed in a more detailed manner in lecture 3.

1.11 The Relative Entropy Method: General Principle

In the regime of *inviscid* hydrodynamic limits, entropy production does **not** balance streaming in the Boltzmann equation. Therefore, the velocity averaging method cannot be applied in the case of inviscid limits, in general[1].

For this reason, we choose another approach, namely to use the regularity of the solution of the target equation together with the relaxation towards local equilibrium to prove the compactness of fluctuations.

Our starting point is to pick u, a smooth solution of the target equations — say, in the case the incompressible Euler equations — and to study the evolution of the quantity

$$Z_\varepsilon(t) := \frac{1}{\delta_\varepsilon^2} H\big(F_\varepsilon | \mathscr{M}_{(1,\delta_\varepsilon u(\varepsilon\delta_\varepsilon t,\varepsilon x),1)}\big)\,.$$

Notice the subtle difference with the usual Boltzmann H Theorem used in the DiPerna-Lions existence theorem of renormalized solutions described above. In the

[1] This is not completely true, however, since the velocity averaging method is at the heart of the kinetic formulation of hyperbolic conservation laws. Unfortunately, while this approach is rather successful in the case of scalar conservation laws, it seems so far limited to some very special kind of hyperbolic system: see P.-L. Lions-B. Perthame-E. Tadmor [57], P.-E. Jabin-B. Perthame [48], B. Perthame [64]

present case, the relative entropy is computed with respect to the *local* Maxwellian equilibrium whose parameters are defined in terms of the solution of the target equation. In the work of DiPerna-Lions, the relative entropy is defined with respect to the *global* Maxwellian equilibrium M.

The idea of studying the evolution of this quantity goes back to the work of H.T. Yau (for Ginzburg-Landau lattice models [78]). It was later adapted to the case of the Boltzmann equation (see chapter 2 in [15] and [56]).

At the formal level, assuming the incompressible Euler scaling, one finds that

$$\begin{aligned}\dot{Z}_\varepsilon(t) = &-\frac{1}{\delta_\varepsilon^2}\int_{\mathbf{R}^3}\nabla_x u : \int_{\mathbf{R}^3}(v-\delta_\varepsilon u)^{\otimes 2}F_\varepsilon dvdx\\ &+\frac{1}{\delta_\varepsilon}\int_{\mathbf{T}^3}\nabla_x p\cdot\int_{\mathbf{R}^3}(v-\delta_\varepsilon u)F_\varepsilon dvdx\,.\end{aligned}$$

The second term on the right hand side vanishes with ε since one expects that

$$\frac{1}{\delta_\varepsilon}\int_{\mathbf{R}^3} vF_\varepsilon\left(\frac{t}{\delta_\varepsilon\varepsilon},\frac{x}{\varepsilon},v\right)dv \to \text{ divergence free field.}$$

The key step in the relative entropy method is to estimate the first term in the right hand side by Z_ε plus $o(1)$, at least locally in time. In other words, for all $T>0$, there exists $C_T>0$ such that

$$\frac{1}{\delta_\varepsilon^2}\iint_{\mathbf{R}^3\times\mathbf{R}^3}\left|\nabla_x u:(v-\delta_\varepsilon u)^{\otimes 2}F_\varepsilon\right| dvdxds \le C_T Z_\varepsilon(t)+o(1)$$

for each $t\in[0,T]$.

Applying Gronwall's lemma, we conclude that

$$Z_\varepsilon(t)\le e^{C_T t}(Z_\varepsilon(0)+o(1))$$

for all $t\in[0,T]$.

By choosing appropriately the initial distribution function $F_\varepsilon\big|_{t=0}$, the right hand side of this inequality vanishes as $\varepsilon\to 0$, and this shows that $Z_\varepsilon(t)\to 0$ as $\varepsilon\to 0$ for all $t>0$. Since the relative entropy $H(F|G)$ somehow measures the "distance" between the distribution functions F and G, this last estimate is exactly what is needed to conclude that the fluctuations of velocity field appropriately scaled

$$\frac{1}{\delta_\varepsilon}\int_{\mathbf{R}^3} vF_\varepsilon\left(\frac{t}{\delta_\varepsilon\varepsilon},\frac{x}{\varepsilon},v\right)dv$$

converge strongly to the solution u of the incompressible Euler equations as $\varepsilon\to 0$.

As we shall see, the constant C_T is (essentially) given by the formula

$$C_T=\|\nabla_x u\|_{L^\infty([0,T]\times\mathbf{R}^3)}$$

and this is precisely why the regularity of solution of the target equation — of the incompressible Euler equation in the present case — is essential for this method.

More precisely, a distinctive feature of the relative entropy method is that it particularly well adapted to study hydrodynamic limits of *weak* (or even renormalized) solutions of kinetic models when the target solution is *smooth* — or at least satisfies some stability property.

2 Lecture 2: The Incompressible Euler Limit

This lecture is devoted to a simplified variant of L. Saint-Raymond's theorem (Theorem 1.9). In order to alleviate the technicalities in the proof, we have chosen to discuss the incompressible Euler limit of the BGK , instead of the Boltzmann equation. As we shall explain below, the BGK equation is a much simplified analogue of the Boltzmann equation.

2.1 The Incompressible Euler Equations

Since the stability of the target solution of the incompressible Euler equation is essential for applying the relative entropy method, we first briefly review the existence, uniqueness and regularity theory for that equation.

The incompressible Euler equation considered here describes the motion of an incompressible fluid, with constant density 1, in space dimension $N=2$ or $N=3$. The state of the fluid at time t is defined by the velocity field $u \equiv u(t,x) \in \mathbf{R}^N$ and the pressure $p \equiv p(t,x) \in \mathbf{R}$. They satisfy the system of partial differential equations (see for instance [55])

$$\begin{aligned} \operatorname{div}_x u = 0\,, \qquad & \text{(continuity equation)} \\ \partial_t u + (u\cdot\nabla_x)u + \nabla_x p = 0\,. \qquad & \text{(momentum equation)} \end{aligned}$$

In the case of an incompressible fluid without external force (such as gravity), the kinetic energy is a locally conserved quantity. Taking the inner product of both sides of the momentum equation above with u leads to the identity:

$$\partial_t(\tfrac{1}{2}|u|^2) + \operatorname{div}_x\left(u(\tfrac{1}{2}|u|^2 + p)\right) = 0\,.$$

(Indeed, one has

$$(u\cdot\nabla_x u + \nabla_x p)\cdot u = u\cdot\nabla_x(\tfrac{1}{2}|u|^2) + u\cdot\nabla_x p = \operatorname{div}_x\left(u(\tfrac{1}{2}|u|^2 + p)\right)$$

because $\operatorname{div}_x u = 0$.)

Another quantity of paramount importance in the theory of inviscid incompressible fluids with constant density is the vorticity field, denoted by Ω, whose evolution is described as follows:

•if $N = 2$, the vorticity field is defined as $\Omega := \partial_1 u_2 - \partial_2 u_1 \in \mathbf{R}$ and one easily checks that

$$\partial_t \Omega + u \cdot \nabla_x \Omega = 0;$$

•if $N = 3$, the vorticity field is defined as $\Omega := \mathrm{curl}_x u \in \mathbf{R}^3$ and one has

$$\partial_t \Omega + (u \cdot \nabla_x)\Omega - (\Omega \cdot \nabla_x)u = 0.$$

2.1.1 Existence and Uniqueness Theory for the Incompressible Euler Equation

Consider the Cauchy problem for the incompressible Euler equations:

$$\begin{cases} \mathrm{div}_x u = 0, \\ \partial_t u + (u \cdot \nabla_x)u + \nabla_x p = 0, \quad x \in \mathbf{R}^N, \\ u\big|_{t=0} = u^{in}. \end{cases}$$

Theorem 2.1 (V. Yudovich, T. Kato) *Consider the Cauchy problem for the incompressible Euler equations in space dimension $N = 2$ or 3. Then*

•$N = 2$*: if $u^{in} \in L^2 \cap C^{1,\alpha}(\mathbf{R}^2)$ for $\alpha \in (0,1)$ and $\Omega^{in} \in L^\infty(\mathbf{R}^2)$, then there exists a unique solution $u \in C(\mathbf{R}_+^*; L^2 \cap C^{1,\alpha}(\mathbf{R}^2))$ of the Cauchy problem for the incompressible Euler equation with initial velocity field u^{in}, and $\Omega \in L^\infty(\mathbf{R}_+ \times \mathbf{R}^2)$;*

•$N = 3$*: if $u^{in} \in L^2 \cap C^{1,\alpha}(\mathbf{R}^3)$ for $\alpha \in (0,1)$, there exists $T^* > 0$ and a unique maximal solution $u \in C([0,T^*); L^2 \cap C^{1,\alpha}(\mathbf{R}^3))$ of the Cauchy problem for the incompressible Euler equation with initial velocity field u^{in}.*

See Theorem 4.1 in [55] for the case $N = 2$, and section 4.3 in the same references for the case $N = 3$. Whether $T^* = +\infty$ in the case where $N = 3$ remains an outstanding open question at the time of this writing.

2.2 Dissipative Solutions of the Incompressible Euler Equation

Since little is known about the global existence of classical solutions of the incompressible Euler equation in space dimension $N = 3$, there have been several attempts at constructing weak solutions of this equation. Weak solutions of the Euler equations are not expected to be unique — in fact, these solutions have some rather paradoxical features (see [71, 72, 22]). Other notions of generalized solutions of the Euler equation have been proposed ([24]). While not much can be said of these solutions, returning to the variational formulation of the incompressible Euler equations viewed as defining a geodesic flow in infinite dimension ([3, 4]) leads to well-posed problems for these equations — but unfortunately, these problems, although interesting in their own right, are different from the Cauchy problem ([16]).

In view of all these difficulties, P.-L. Lions proposed a very weak notion of solution of the incompressible Euler equation, which he called "dissipative solutions", and whose definition is recalled below (see section 4.4 in [55]).

Set

$$\begin{aligned}\mathscr{X}_T := \{v \in C([0,T];L^2(\mathbf{R}^3)) \text{ s.t. } \operatorname{div}_x v = 0\,,\ \Sigma(v) \in L^1([0,T];L^\infty(\mathbf{R}^3))\\ \text{and } E(v) \in L^1([0,T];L^2(\mathbf{R}^3))\}\end{aligned}$$

where

$$\Sigma(v) := \nabla_x v + (\nabla_x v)^T\,, \quad \text{and } E(v) := \partial_t v + (v\cdot\nabla_x)v\,.$$

Definition 2.2 (P.-L. Lions) *A vector field*[2] $u \in C_b(\mathbf{R}_+; w-L^2(\mathbf{R}^3))$ *is a dissipative solution of the Cauchy problem for the incompressible Euler equation with initial velocity field* u^{in} *if* $\operatorname{div}_x u = 0$ *and, for each* $T > 0$*, each* $v \in \mathscr{X}_T$ *and each* $t \in [0,T]$*, one has*

$$\begin{aligned}\tfrac12\|u-v\|^2_{L^2}(t) \le \exp\left(\int_0^t 2\|\Sigma(v)\|_{L^\infty}(s)ds\right)\tfrac12\|u^{in} - v\big|_{t=0}\|^2_{L^2}\\ + \int_0^t \exp\left(\int_\tau^t 2\|\Sigma(v)\|_{L^\infty}(s)ds\right)\int E(v)\cdot(u-v)(\tau,x)dxd\tau\,.\end{aligned}$$

The nicest features of this notion of dissipative solution is that the Cauchy problem for the incompressible Euler equation always has at least one dissipative solution, and also the fact that classical solutions of the incompressible Euler equation are uniquely determined by their initial data within the class of dissipative solutions.

Theorem 2.3 (P.-L. Lions [55]) *For each* $u^{in} \in L^2(\mathbf{R}^N)$ *s.t.* $\operatorname{div}_x u^{in} = 0$*, there exists a dissipative solution of the Cauchy problem for the incompressible Euler equation defined for all* $t \ge 0$*. Besides*

•*if* $u \in C^1_b([0,T]\times\mathbf{R}^N)$ *is a classical solution of the Cauchy problem for the Euler equation with initial velocity field* u^{in}*, then u is a dissipative solution.*

•*if the Cauchy problem for the incompressible Euler equation with initial velocity field* u^{in} *has a solution* $\overline{u} \in \mathscr{X}_T$ *for some* $T > 0$*, any dissipative solution u of the incompressible Euler equation with initial velocity field* u^{in} *satisfies*

$$u(t,x) = \overline{u}(t,x) \text{ for a.e. } x \in \mathbf{R}^N\,,\ \text{ for all } t \in [0,T]$$

Proof. Observe that limit points of Leray solutions of the incompressible Navier-Stokes equation in the vanishing viscosity limit are dissipative solutions of the incompressible Euler equations. This implies the global existence of dissipative solutions of the Cauchy problem for the incompressible Euler equation for all initial square integrable, divergence free velocity field u^{in}.

[2] The notation $w-L^p(X)$ designates the Lebesgue space $L^p(X)$ endowed with its weak topology.

Observe next that, if u is a C^1 solution of the Euler equation

$$E(u) - E(v) = (\partial_t + u \cdot \nabla_x)(u - v) + (u - v) \cdot \nabla_x v$$

which implies that

$$(\partial_t + u \cdot \nabla_x)\tfrac{1}{2}|u - v|^2 + \Sigma(v) : (u - v)^{\otimes 2} = (E(u) - E(v)) \cdot (u - v)\,.$$

Since $\operatorname{div}_x u = 0$, integrating in x both sides of the identity above shows that

$$\frac{d}{dt}\tfrac{1}{2}\|u - v\|_{L^2}^2 \leq \|\Sigma(v)\|_{L^\infty}\|u - v\|_{L^2}^2 + (E(v)|u - v)_{L^2}$$

since

$$\int_{\mathbf{R}^3} E(u) \cdot (u - v)dx = -\int_{\mathbf{R}^3} \nabla_x p \cdot (u - v)dx = \int_{\mathbf{R}^3} p \operatorname{div}_x(u - v)dx = 0\,.$$

Applying Gronwall's lemma shows that u is a dissipative solution of Euler's equation.

Finally the last property, usually referred to as the "weak-strong uniqueness" property of dissipative solutions of the incompressible Euler equation is obtained by the observation below. If one choose $v = \overline{u}$ in the defining inequality for dissipative solutions, one finds that

$$\int_{\mathbf{R}^3} E(v) \cdot (u - v)(\tau, x)dx = -\int_{\mathbf{R}^3} \nabla_x \overline{p} \cdot (u - \overline{u})(\tau, x)dx = 0$$

because $\operatorname{div}_x u = \operatorname{div}_x \overline{u} = 0$. Therefore

$$\tfrac{1}{2}\|u - \overline{u}\|_{L^2}^2(t) \leq \exp\left(\int_0^t 2\|\Sigma(\overline{u})\|_{L^\infty}(s)ds\right)\tfrac{1}{2}\|u^{in} - \overline{u}\big|_{t=0}\|_{L^2}^2 = 0$$

for all $t \in [0, T]$.

Of course, it is unknown whether two dissipative solutions of the incompressible Euler equation with the same initial condition coincide on the time interval on which they are both defined.

Any dissipative solution of the Euler equation that is obtained as limits points of Leray solutions of the Navier-Stokes equation in the vanishing viscosity limit satisfies the following variant of the motion equation:

$$\partial_t u + \operatorname{div}_x(u \otimes u + \sigma) + \nabla_x p = 0\,,$$

where $\sigma \equiv \sigma(t, x) \in M_3(\mathbf{R})$ is a matrix field satisfying

$$\sigma = \sigma^T \geq 0\,.$$

Whether $\sigma = 0$ — in other words, whether u is a solution of the Euler equation in the sense of distributions — remains unknown at the time of this writing.

2.3 The BGK Model with Constant Relaxation Time

In order to alleviate some technical steps in the proof of the incompressible Euler limit of the Boltzmann equation, we shall consider as our starting point the BGK model with constant relaxation time instead of the Boltzmann equation itself. Some of the unpleasant features of the theory of renormalized solutions of the Boltzmann equation, especially regarding the local conservation laws either disappear or become significantly simpler with the BGK model.

The idea is therefore to replace the Boltzmann equation with the simplest imaginable relaxation model with constant relaxation time $\tau > 0$

$$(\partial_t + v \cdot \nabla_x)F = \frac{1}{\tau}(M_F - F), \quad x \in \mathbf{T}^3, \ v \in \mathbf{R}^3,$$

where

$$M_F \equiv M_F(t,x,v) := \mathscr{M}_{(\rho_F, u_F, \theta_F)(t,x)}(v),$$

with

$$\int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} M_F(t,x,v)dv = \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} F(t,x,v)dv.$$

In other words, (ρ_F, u_F, θ_F) are defined as follows:

$$\rho_F = \int_{\mathbf{R}^3} F dv, \quad u_F = \frac{1}{\rho_F}\int_{\mathbf{R}^3} vF dv, \quad \theta_F = \frac{1}{\rho_F}\int_{\mathbf{R}^3} \tfrac{1}{3}|v - u_F|^2 F dv.$$

This model Boltzmann equation is called the "BGK model", after Bhatnagar, Gross and Krook, who proposed (a more complicated variant of) this model for the first time in 1954 [14].

We recall below the notation already adopted above for Maxwellians: in space dimension 3, for $\rho \geq 0$, $u \in \mathbf{R}^3$ and $\theta > 0$,

$$\mathscr{M}_{(\rho,u,\theta)}(v) := \frac{\rho}{(2\pi\theta)^{3/2}} e^{-|v-u|^2/2\theta}.$$

In the limit as $\theta \to 0^+$, one has $\mathscr{M}_{(\rho,u,\theta)} \to \mathscr{M}_{(\rho,u,0)}$, where

$$\mathscr{M}_{(\rho,u,0)} := \rho\,\delta(v-u).$$

In the particular case $\rho = \theta = 1$ and $u = 0$, we denote as above

$$M(v) := \mathscr{M}_{(1,0,1)}(v) = \tfrac{1}{(2\pi)^{3/2}} e^{-|v|^2/2}.$$

2.4 Formal Properties of the BGK Model

Classical solutions of the BGK model satisfy exactly the same local conservation laws of mass, momentum and energy as classical solutions of the Boltzmann equation, under appropriate decay assumptions as $|v| \to \infty$.

Proposition 2.4 *Let $F \in C(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$ such that $\nabla_{t,x}F \in C(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$ satisfy*

$$F \geq 0 \text{ and } \sup_{t+|x| \leq R} F(t,x,v) + |\nabla_{t,x}F(t,x,v)| \leq \frac{C_R}{(1+|v|)^7}$$

for each $R > 0$. Then

$$\partial_t \int_{\mathbf{R}^3} F dv + \operatorname{div}_x \int_{\mathbf{R}^3} vF dv = 0\,, \quad \text{(mass)}$$

$$\partial_t \int_{\mathbf{R}^3} vF dv + \operatorname{div}_x \int_{\mathbf{R}^3} v^{\otimes 2} F dv = 0\,, \quad \text{(momentum)}$$

$$\partial_t \int_{\mathbf{R}^3} \tfrac{1}{2}|v|^2 F dv + \operatorname{div}_x \int_{\mathbf{R}^3} v\tfrac{1}{2}|v|^2 F dv = 0\,. \quad \text{(energy)}$$

Proof. The assumptions on the decay of F and $\nabla_{t,x}F$ as $|v| \to \infty$ imply that

$$\partial_t \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F dv + \operatorname{div}_x \int_{\mathbf{R}^3} \begin{pmatrix} v \\ v \otimes v \\ \frac{1}{2}v|v|^2 \end{pmatrix} F dv = \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} (\partial_t + v \cdot \nabla_x) F dv$$
$$= \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} (M_F - F) dv = 0\,,$$

by definition of M_F.

They also satisfy the following local variant of Boltzmann's H Theorem.

Proposition 2.5 *Let $F \in C(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$ such that $\nabla_{t,x}F \in C(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$ satisfy*

$$F \geq 0 \text{ and } \sup_{t+|x| \leq R} (F \ln F(t,x,v) + |\nabla_{t,x}(F \ln F)(t,x,v)|) \leq \frac{C_R}{(1+|v|)^4}$$

for each $R > 0$. Then

$$\partial_t \int_{\mathbf{R}^3} F \ln F dv + \operatorname{div}_x \int_{\mathbf{R}^3} vF \ln F dv = \frac{1}{\tau} \int_{\mathbf{R}^3} (M_F - F) \ln \frac{F}{M_F} dv \leq 0\,.$$

Proof. Indeed $\ln M_F$ is a linear combination of $1, v_1, v_2, v_3, |v|^2$ so that

$$\int_{\mathbf{R}^3} (F - M_F) \ln M_F dv = 0\,,$$

again by definition of M_F.

2.5 The Cauchy Problem for the BGK Model with Constant Relaxation Time

Consider the Cauchy problem

$$\begin{cases} (\partial_t + v\cdot\nabla_x)F = \dfrac{1}{\tau}(M_F - F)\,, \quad x\in\mathbf{T}^3\,, v\in\mathbf{R}^3\,, \\ F\big|_{t=0} = F^{in}\,. \end{cases} \tag{9}$$

Theorem 2.6 (B. Perthame-M. Pulvirenti) *Assume that there exists $\rho_2 > \rho_1 > 0$ and $\theta_2 > \theta_1 > 0$ such that the initial distribution function F^{in} satisfies the inequalities*

$$\mathscr{M}_{(\rho_1,0,\theta_1)} \le F^{in} \le \mathscr{M}_{(\rho_2,0,\theta_2)}\,.$$

Then there exists a unique solution of the Cauchy problem (9), which satisfies

$$C_1(t,\tau) \le \rho_F(t,x), \theta_F(t,x) \le C_2(t,\tau)\,, \quad |u_F(t,x)| \le C_2(t,\tau)\,,$$
$$\text{and } \sup_{x,v} |v|^m F(t,x,v) < C_3(t,\tau,m)\,.$$

See [65] for a proof of this result.

Since the relaxation time in the model above is a constant, the collision term $M_F - F$ is homogeneous of degree 1 in the distribution function F. (In other words, one has $M_{\lambda F} - \lambda F = \lambda(M_F - F)$. This is precisely the reason why there is no need for the renormalization procedure used for the Boltzmann equation. Thus the existence theory is significantly simpler for this model than for the Boltzmann equation itself.

In fact, the genuine BGK model involves a relaxation time that is proporttional to the reciprocal local macroscopic density. In other words, this model is of the form

$$(\partial_t + v\cdot\nabla_x)F = \frac{1}{\tau_0}\rho_F(M_F - F)\,,$$

with

$$\rho_F(t,x) := \int_{\mathbf{R}^3} F(t,x,v)dv\,.$$

The collision term $\frac{1}{\tau_0}\rho_F(M_F - F)$ is now homogeneous of degree 2, meaning that

$$\rho_{\lambda F}(M_{\lambda F} - \lambda F) = \lambda^2 \rho_F(M_F - F)\,,$$

just like the Boltzmann collision integral which is a quadratic operator. This model is obviously more natural than the one with constant relaxation time, since the higher the local density ρ_F, the smaller the local particle mean free path, i.e. τ_0/ρ_F. This

BGK model is used as a toy model in rarefied gas dynamics. Unfortunately, even though the numerical analysis of the BGK model is significantly simpler than that of the Boltzmann equation, much less is known on the mathematical analysis of this model than on the Boltzmann equation itself. For instance, the renormalization procedure is rather uneffective on the BGK model, so that there is no analogue of the DiPerna-Lions theory on that model.

2.6 The BGK Equation in the Incompressible Euler Scaling

Set the relaxation time $\tau = \varepsilon^q$ with $q > 1$ for $\varepsilon > 0$ small enough, and rescale time variable as $\hat{t} = t/\varepsilon$. The Cauchy problem for the BGK equation with constant relaxation time takes the form

$$\begin{cases} (\varepsilon\partial_{\hat{t}}t + v\cdot\nabla_x)F_\varepsilon = \dfrac{1}{\varepsilon^q}(M_{F_\varepsilon} - F_\varepsilon)\,, \quad x \in \mathbf{T}^3\,,\ v \in \mathbf{R}^3\,, \\ F\big|_{t=0} = \mathscr{M}_{(1,\varepsilon u^{in},1)}\,. \end{cases} \tag{10}$$

Henceforth, we assume that

$$u^{in} \in C(\mathbf{T}^3)\,, \qquad \text{with } \operatorname{div} u^{in} = 0\,.$$

The incompressible Euler limit of the BGK model with constant relaxation time is described in the following theorem.

Theorem 2.7 (L. Saint-Raymond) *Let $u^{in} \in C^{1,\alpha}(\mathbf{T}^N)$ be s.t. $\operatorname{div} u^{in}$ and let u be the maximal solution of the incompressible Euler equation with initial data u^{in} defined on $[0,T^*)$. Let F_ε be the solution of the scaled BGK equation with initial data $\mathscr{M}_{(1,\varepsilon u^{in},1)}$. Then*

$$\frac{1}{\varepsilon}\int_{\mathbf{R}^N} vF_\varepsilon(t,\cdot,v)dv \to u(t,\cdot) \text{ in weak } L^1(\mathbf{T}^N)\,,$$

uniformly on $[0,T]$ for each $0 \le T < T^*$ as $\varepsilon \to 0$.

The proof of this theorem can be found in [66]. This result was later extended to renormalized solutions of the Boltzmann equation [68]. Earlier earlier partial results were obtained by Golse [15], and by P.-L. Lions and N. Masmoudi [56].

This result is based on the relative entropy method, which is a very important tool in the rigorous asymptotic analysis of partial differential equations. For that reason, we have given a rather detailed account of the proof in the case of the BGK model. Proving the same result for the Boltzmann equation involves additional technicalities that are special to the theory of renormalized solutions.

2.7 Proof of the Incompressible Euler Limit

This section is devoted to L. Saint-Raymond's proof of the incompressible Euler limit of the BGK equation.

2.7.1 Step 1: Uniform Estimates

All the uniform estimates on this problem come from (the analogue of) Boltzmann's H theorem. Specifically, we compute the evolution of the relative entropy $H(F_\varepsilon|M)$; one has

$$\varepsilon\partial_t \int_{\mathbf{R}^3} \left(F_\varepsilon \ln\left(\frac{F_\varepsilon}{M}\right) - F_\varepsilon + M\right) dv + \operatorname{div}_x \int_{\mathbf{R}^3} v \left(F_\varepsilon \ln\left(\frac{F_\varepsilon}{M}\right) - F_\varepsilon + M\right) dv$$
$$= \frac{1}{\varepsilon^q} \int_{\mathbf{R}^3} (M_{F_\varepsilon} - F_\varepsilon) \ln \frac{F}{M_{F_\varepsilon}} dv \leq 0\,,$$

in view of the decay (in $|v|$) estimate in the Perthame-Pulvirenti theorem. Integrating further in t and x, one finds that

$$H(F_\varepsilon|M)(t) + \frac{1}{\varepsilon^{q+1}} \int_0^\infty D(F_\varepsilon) dt = H(\mathscr{M}_{(1,\varepsilon u^{in},1)}|M) = \tfrac{1}{2}\varepsilon^2 \|u^{in}\|_{L^2}^2\,,$$

so that

$$H(F_\varepsilon|M)(t) \leq C^{in}\varepsilon^2\,, \quad \text{and} \int_0^\infty D(F_\varepsilon) dt \leq C^{in}\varepsilon^{q+3}$$

with $C^{in} = \frac{1}{2}\|u^{in}\|_{L^2}^2$.

Instead of the distribution function F_ε itself, it will be more convenient to work with the relative fluctuation thereof, denoted

$$g_\varepsilon := \frac{F_\varepsilon - M}{\varepsilon M}\,.$$

Consider the function h defined on $(-1,\infty)$ by the formula

$$h(z) := (1+z)\ln(1+z) - z\,.$$

Its Legendre dual, henceforth denoted h^*, is given by the formula

$$h^*(y) := e^y - y - 1\,, \quad y \geq 0\,.$$

The Young inequality for the convex function h implies that, for all $\alpha > \varepsilon$,

$$\tfrac{1}{4}(1+|v|^2)|g_\varepsilon| = \frac{\alpha}{\varepsilon^2}\tfrac{1}{4}\frac{\varepsilon}{\alpha}(1+|v|^2)\varepsilon|g_\varepsilon|$$
$$\leq \frac{\alpha}{\varepsilon^2} h(\varepsilon|g_\varepsilon|) + \frac{\alpha}{\varepsilon^2} h^*\left(\frac{\varepsilon}{\alpha}\tfrac{1}{4}(1+|v|^2)\right)\,.$$

Using the elementary inequalities

$$h(|z|) \leq h(z)\,, \text{ and } h^*(\theta y) = \sum_{k\geq 2} \frac{\theta^k z^k}{k!} \leq \theta^2 \sum_{k\geq 2} \frac{z^k}{k!} = \theta^2 h^*(y)$$

for all $z > -1$ and all $y \geq 0$ whenever $0 \leq \theta \leq 1$, we conclude that

$$\begin{aligned}\tfrac{1}{4}(1+|v|^2)|g_\varepsilon| &= \frac{\alpha}{\varepsilon^2}\tfrac{1}{4}\frac{\varepsilon}{\alpha}(1+|v|^2)\varepsilon|g_\varepsilon| \\ &\leq \frac{\alpha}{\varepsilon^2}h(\varepsilon g_\varepsilon) + \frac{1}{\alpha}h^*\left(\tfrac{1}{4}(1+|v|^2)\right)\,.\end{aligned}$$

A first major consequence of the uniform bounds obtained above is the next proposition.

Proposition 2.8 *The family* $(1+|v|^2)g_\varepsilon$ *is weakly relatively compact in the space* $L^1([0,T];L^2(\mathbf{T}^3\times\mathbf{R}^3, Mdvdx))$ *for all* $T>0$. *If* $(1+|v|^2)g$ *is a limit point of this family (along a sequence* $\varepsilon_n \to 0$*), then*

$$\iint_{\mathbf{T}^3\times\mathbf{R}^3} g^2 M dv dx \leq \underline{\lim}\, \frac{1}{\varepsilon_n^2} H(F_{\varepsilon_n}|M)\,.$$

Another important observation is the following lemma, which follows from the elementary inequality

$$(1+z)\ln(1+z) - z \leq z\ln(1+z)\,, \quad z > -1\,.$$

Lemma 2.9 *For each* $\varepsilon > 0$ *and all* $t \geq 0$,

$$H(F_\varepsilon|M_{F_\varepsilon})(t) \leq D(F_\varepsilon)(t)\,.$$

2.7.2 Step 2: the Modulated Relative Entropy

First observe that, for each vector field $u \in L^2(\mathbf{T}^3)$, one has

$$H(F_\varepsilon|\mathscr{M}_{1,u,\theta}) = H(F_\varepsilon|M_{F_\varepsilon}) + H(M_{F_\varepsilon}|\mathscr{M}_{1,u,\theta})\,,$$

since M_{F_ε} and M have the same total mass.

Let $w \equiv w(t,x) \in \mathbf{R}^3$ be a vector field of class C^1 on $[0,T]\times\mathbf{T}^3$ satisfying the incompressibility $\operatorname{div}_x w = 0$, but not necessarily a solution of the Euler equation. Then

$$
\begin{aligned}
H(F_\varepsilon|\mathscr{M}_{(1,\varepsilon w,1)}) &= H(F_\varepsilon|M) + \iint_{\mathbf{T}^3\times\mathbf{R}^3} F_\varepsilon \ln\left(\frac{M}{\mathscr{M}_{(1,\varepsilon w,1)}}\right) dxdv \\
&= H(F_\varepsilon|M) + \iint_{\mathbf{T}^3\times\mathbf{R}^3} \tfrac{1}{2}(|v-\varepsilon w|^2 - |v|^2) F_\varepsilon dxdv \\
&= H(F_\varepsilon|M) + \iint_{\mathbf{T}^3\times\mathbf{R}^3} (\tfrac{1}{2}\varepsilon^2|w|^2 - \varepsilon v\cdot w) F_\varepsilon dxdv \\
&= H(F_\varepsilon|M) + \int_{\mathbf{T}^3} \rho_{F_\varepsilon}(\tfrac{1}{2}\varepsilon^2|w|^2 - \varepsilon u_{F_\varepsilon}\cdot w) dx\,.
\end{aligned}
$$

Apply first the local conservation laws implied by the BGK equation

$$
\begin{aligned}
\varepsilon\partial_t \rho_{F_\varepsilon} + \operatorname{div}_x(\rho_{F_\varepsilon} u_{F_\varepsilon}) &= 0\,, \\
\varepsilon\partial_t(\rho_{F_\varepsilon} u_{F_\varepsilon}) + \operatorname{div}_x \int_{\mathbf{R}^3} v^{\otimes 2} F_\varepsilon dv &= 0\,.
\end{aligned}
$$

Using the operator E entering the definition of dissipative solutions, one has

$$
\partial_t w = E(w) - (w\cdot\nabla_x)w\,,
$$

and therefore

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbf{T}^3} \rho_{F_\varepsilon}(\tfrac{1}{2}\varepsilon^2|w|^2 - \varepsilon u_{F_\varepsilon}\cdot w) dx \\
&= \iint_{\mathbf{T}^3\times\mathbf{R}^3} \left(\nabla_x w : (v-\varepsilon w)^{\otimes 2} - \varepsilon E(w)\cdot(v-\varepsilon w)\right) F_\varepsilon dxdv \\
&= \iint_{\mathbf{T}^3\times\mathbf{R}^3} \left(\Sigma(w) : (v-\varepsilon w)^{\otimes 2} - \varepsilon E(w)\cdot(v-\varepsilon w)\right) F_\varepsilon dxdv\,.
\end{aligned}
$$

The core of the proof is the inequality stated in the next proposition.

Proposition 2.10 *Let $u^{in} \in C(\mathbf{T}^3)$ satisfy $\operatorname{div} u^{in} = 0$; then for each test vector field $w \in C^1([0,T]\times\mathbf{T}^3;\mathbf{R}^3)$ such that $\operatorname{div}_x w = 0$, one has*

$$
\begin{aligned}
\frac{1}{\varepsilon^2} H(F_\varepsilon|\mathscr{M}_{(1,\varepsilon w,1)})(t) + \frac{1}{\varepsilon^{3+q}}\int_0^t D(F_\varepsilon) ds &\le \tfrac{1}{2}\|u^{in} - w\big|_{t=0}\|_{L^2}^2 \\
&- \frac{1}{\varepsilon^2}\int_0^t \iint_{\mathbf{T}^3\times\mathbf{R}^3} \Sigma(w) : (v-\varepsilon w)^{\otimes 2} F_\varepsilon dxdv \\
&- \frac{1}{\varepsilon}\int_0^t \iint_{\mathbf{T}^3\times\mathbf{R}^3} E(w)\cdot(v-\varepsilon w) F_\varepsilon dxdv\,.
\end{aligned}
$$

This inequality is the analogue for the BGK equation of the weak-strong uniqueness inequality for the Euler equation, i.e.

$$
\begin{aligned}
&\tfrac{1}{2}\|u-w\|_{L^2}^2(t) \le \tfrac{1}{2}\|u^{in} - w\big|_{t=0}\|_{L^2}^2 \\
&+ \int_0^t \|\Sigma(w)\|_{L^\infty}\|u-w\|_{L^2}^2(s) ds + \int_0^t (E(v)|u-v)_{L^2}(s) ds\,,
\end{aligned}
$$

leading to the notion of dissipative solution (after applying Gronwall's inequality).

More precisely, one has the following correspondences

•Velocity field

$$\frac{1}{\varepsilon}\int_{\mathbf{R}^3} vF_\varepsilon dv \leftrightarrow u\,,$$

•Modulated energy

$$\frac{1}{\varepsilon^2}H(F_\varepsilon|\mathscr{M}_{(1,\varepsilon w,1)})(t) \leftrightarrow \tfrac{1}{2}\|u-w\|_{L^2}^2(t)\,,$$

•Modulated inertial term

$$\frac{1}{\varepsilon^2}\int_0^t\iint_{\mathbf{T}^3\times\mathbf{R}^3}\Sigma(w):(v-\varepsilon w)^{\otimes 2}F_\varepsilon dxdv$$
$$\leftrightarrow \int_0^t \|\Sigma(w)\|_{L^\infty}\|u-w\|_{L^2}^2(s)ds\,.$$

It remains to control both terms on the right hand side of the inequality in the proposition above in terms of the relative entropy and to conclude by Gronwall's lemma.

The last such term is disposed of without difficulty. We already know that

$$(1+|v|^2)g_\varepsilon \to (1+|v|^2)g \quad \text{weakly in } L^1([0,T];L^1(\mathbf{T}^3\times\mathbf{R}^3;Mdxdv))\,,$$

with

$$g\in L^\infty([0,T];L^2(\mathbf{T}^3\times\mathbf{R}^3;Mdxdv))\,.$$

Therefore

Lemma 2.11 *Let* $U:=\langle vg\rangle$*; then* $\operatorname{div}_x U=0$ *and*

$$\frac{1}{\varepsilon}\int_0^t\iint_{\mathbf{T}^3\times\mathbf{R}^3}E(w)\cdot(v-\varepsilon w)F_\varepsilon dxdv \to \int_{\mathbf{T}^3}E(w)\cdot(U-w)dx$$

weakly in $L^1([0,T])$ *for all* $T>0$.

2.7.3 Step 3: Controling the Modulated Inertial Term

In the case of the Euler equation, the contribution of the inertial term to the energy balance, i.e. $\Sigma(v):(u-v)^{\otimes 2}$, is obviously controlled as follows:

$$|\Sigma(v):(u-v)^{\otimes 2}|\le \|\Sigma(v)\|_{L^\infty}\|u-v\|_{L^2}^2\,.$$

Whether the analogue of the modulated inertial term in the context of the BGK equation can be controlled by the modulated relative entropy is more subtle. A major difficulty in obtaining this type of control is the fact that the relative entropy is subquadratic, unless the fluctuations of distribution function are already known to be small (of order ε).

However, this difficulty can be solved by using the entropy production as well as the relative entropy. This control is explained in the next lemma, which is the key argument in the proof.

Lemma 2.12 *Under the same assumptions as in Theorem 2.7,*

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_0^t \iint_{\mathbf{T}^3\times\mathbf{R}^3} &\Sigma(w):(v-\varepsilon w)^{\otimes 2}F_\varepsilon dxdvds\\ &\le C\|\Sigma(w)\|_{L^\infty}\int_0^t \frac{1}{\varepsilon^2}H(F_\varepsilon|\mathscr{M}_{(1,\varepsilon w,1)})ds\\ &+\varepsilon^{(q-1)/2}\|\Sigma(w)\|_{L^\infty}\frac{1}{\varepsilon^{q+3}}\int_0^t D(F_\varepsilon)ds\\ &+C\varepsilon^{(q-1)/2}\|\Sigma(w)\|_{L^1}\,.\end{aligned}$$

The idea is to split the distribution function F_ε as

$$F_\varepsilon = M_{F_\varepsilon} + (F_\varepsilon - M_{F_\varepsilon})\,,$$

and use both the entropy and entropy production bounds.

Proof (Sketch of the proof). By definition of M_{F_ε}, one has

$$\begin{aligned}&\iint_{\mathbf{T}^3\times\mathbf{R}^3}\Sigma(w):(v-\varepsilon w)^{\otimes 2}M_{F_\varepsilon}dxdv\\ &=\iint_{\mathbf{T}^3}\Sigma(w):\left((u_{F_\varepsilon}-\varepsilon w)^{\otimes 2}+3\theta_{F_\varepsilon}I\right)\rho_{F_\varepsilon}dx\\ &=\int_{\mathbf{T}^3}\Sigma(w):(u_{F_\varepsilon}-\varepsilon w)^{\otimes 2}\rho_{F_\varepsilon}dx\&,.\end{aligned}$$

Notice that the 2nd equality follows from $\operatorname{div}_x w = 0$ so that

$$\operatorname{trace}(\Sigma(w)(t,x)) = \operatorname{div}_x w = 0\,.$$

This term should be compared with

$$\begin{aligned}H(M_{F_\varepsilon}|\mathscr{M}_{(1,\varepsilon w,1)}) := &\int_{\mathbf{T}^3}(\rho_{F_\varepsilon}\ln\rho_{F_\varepsilon}-\rho_{F_\varepsilon}+1)dx\\ &+\tfrac{1}{2}\int_{\mathbf{T}^3}\rho_{F_\varepsilon}|u_{F_\varepsilon}-\varepsilon w|^2dx\\ &+\tfrac{3}{2}\int_{\mathbf{T}^3}\rho_{F_\varepsilon}(\theta_{F_\varepsilon}-\ln\theta_{F_\varepsilon}-1)dx\\ &\le H(F_\varepsilon|\mathscr{M}_{(1,\varepsilon w,1)})\,,\end{aligned}$$

so that

$$\iint_{\mathbf{T}^3\times\mathbf{R}^3}\Sigma(w):(v-\varepsilon w)^{\otimes 2}M_{F_\varepsilon}dxdv \\ \leq 2\|\Sigma(w)\|_{L^\infty}H(F_\varepsilon|\mathscr{M}_{(1,\varepsilon w,1)})\,.$$

At this point, we seek to decompose the space of positions according to whether or not the local hydrodynamic moments are $O(1)$ fluctuations of equilibrium. Specifically

$$\begin{aligned}&\iint_{\mathbf{T}^3\times\mathbf{R}^3}\Sigma(w):(v-\varepsilon w)^{\otimes 2}(F_\varepsilon-M_{F_\varepsilon})dxdv\\ =&\iint_{\mathscr{A}_\varepsilon(t)\times\mathbf{R}^3}\Sigma(w):(v-\varepsilon w)^{\otimes 2}(F_\varepsilon-M_{F_\varepsilon})dxdv\\ +&\iint_{\mathscr{B}_\varepsilon(t)\times\mathbf{R}^3}\Sigma(w):(v-\varepsilon w)^{\otimes 2}(F_\varepsilon-M_{F_\varepsilon})dxdv\,,\end{aligned}$$

where $\mathscr{A}_\varepsilon(t)\subset\mathbf{T}^3$ is defined as the set of xs such that

$$\max(|\rho_{F_\varepsilon}(t,x)-1|,|u_{F_\varepsilon}-\varepsilon w|(t,x),|\theta_{F_\varepsilon}(t,x)-1|)\leq\tfrac12\,,$$

while $\mathscr{B}_\varepsilon(t):=\mathbf{T}^3\setminus\mathscr{A}_\varepsilon(t)$.

On $\mathscr{A}_\varepsilon(t)$

$$\begin{aligned}&\frac{1}{4\varepsilon^2}|v-\varepsilon w|^2\left|\frac{F_\varepsilon}{M_{F_\varepsilon}}-1\right|\\ &\leq\frac{1}{\varepsilon^{(q+7)/2}}h\left(\frac{F_\varepsilon}{M_{F_\varepsilon}}-1\right)+\frac{1}{\varepsilon^{(q+7)/2}}h^*(\varepsilon^{(q+3)/2}|v-\varepsilon w|^2)\\ &\leq\frac{1}{\varepsilon^{(q+7)/2}}\left(\frac{F_\varepsilon}{M_{F_\varepsilon}}-1\right)\ln\left(\frac{F_\varepsilon}{M_{F_\varepsilon}}\right)+\frac{1}{\varepsilon^{(q-1)/2}}h^*(\tfrac14|v-\varepsilon w|^2)\,,\end{aligned}$$

and

$$M_{F_\varepsilon}(t,x,v)\leq\frac{3}{2\pi^{3/2}}e^{-(|v-\varepsilon w|-\frac12)^2/3}\,,$$

so that

$$\iint_{\mathscr{A}_\varepsilon(t)\times\mathbf{R}^3}\Sigma(w):(v-\varepsilon w)^{\otimes 2}(F_\varepsilon-M_{F_\varepsilon})dxdv \\ \leq 4\varepsilon^{(q-1)/2}\|\Sigma(w)\|_{L^\infty}(D(F_\varepsilon)+C_1)\,.$$

On $\mathscr{B}_\varepsilon(t)$

$$\begin{aligned}\int_{\mathbf{R}^3}|v-\varepsilon w|^2F_\varepsilon dv&=\int_{\mathbf{R}^3}|v-\varepsilon w|^2M_{F_\varepsilon}dv\\ &=\rho_{F_\varepsilon}(|u_{F_\varepsilon}-\varepsilon w|^2+3\theta_{F_\varepsilon})\\ &\leq C_2\rho_{F_\varepsilon}(|u_{F_\varepsilon}-\varepsilon w|^2+3(\theta_{F_\varepsilon}-\ln\theta_{F_\varepsilon}-1))\,,\end{aligned}$$

so that

$$\iint_{\mathscr{B}_\varepsilon(t)\times\mathbf{R}^3}\Sigma(w):(v-\varepsilon w)^{\otimes 2}(F_\varepsilon - M_{F_\varepsilon})dxdv$$
$$\leq C_2 H(M_{F_\varepsilon}|\mathscr{M}_{(1,\varepsilon w,1)}) \leq C_2 H(F_\varepsilon|\mathscr{M}_{(1,\varepsilon w,1)})\,.$$

2.7.4 Step 4: Applying Gronwall's Inequality

We start from the identity

$$\begin{aligned}\frac{1}{\varepsilon^2}H(F_\varepsilon|\mathscr{M}_{(1,\varepsilon w,1)})(t) = {} & \frac{1}{\varepsilon^2}H(F_\varepsilon|M)\\ & + \frac{1}{\varepsilon^2}\iint_{\mathbf{T}^3\times\mathbf{R}^3} F_\varepsilon \ln\left(\frac{M}{\mathscr{M}_{(1,\varepsilon w,1)}}\right)dxdv\\ & \leq C^{in} + \tfrac{1}{2}\int_{\mathbf{T}^3}\rho_{F_\varepsilon}|w|^2dx - \frac{1}{\varepsilon}\int_{\mathbf{T}^3}\rho_{F_\varepsilon}u_{F_\varepsilon}\cdot w dx\,,\end{aligned}$$

and use the conservation of mass to check that

$$\tfrac{1}{2}\int_{\mathbf{T}^3}\rho_{F_\varepsilon}|w|^2dx \leq \|w\|^2_{L^\infty}\int_{\mathbf{T}^3}\rho_{F_\varepsilon}dx = \|w\|^2_{L^\infty}\,.$$

The entropy control implies that

$$\frac{1}{\varepsilon}\rho_{F_\varepsilon}u_{F_\varepsilon} = \frac{1}{\varepsilon}\int vF_\varepsilon dv \text{ is bounded in } L^\infty(\mathbf{R}_+;L^1(\mathbf{T}^3))\,.$$

Hence there exists a positive constant C such that

$$\frac{1}{\varepsilon^2}H(F_\varepsilon|\mathscr{M}_{(1,\varepsilon w,1)}) \leq C\,.$$

Therefore, up to extracting a subsequence if needed, one has

$$\frac{1}{\varepsilon^2}H(F_\varepsilon|\mathscr{M}_{(1,\varepsilon w,1)}) \to H_w \text{ in } L^\infty([0,T]) \text{ weak-}*\,,$$

for each $T>0$.

Applying Proposition 2.10 together with Lemmas 2.11 and 2.12 above, one finds that

$$H_w(t) \leq H_w(0) + C\|\Sigma(w)\|_{L^\infty}\int_0^t H_w ds - \int_0^t\int_{\mathbf{T}^3}E(w)\cdot(U-w)dxds\,.$$

Gronwall's inequality implies that

$$
\begin{aligned}
H_w(t) &\leq H_w(0)\exp\left(C\int_0^t \|\Sigma(w)\|_{L^\infty}(s)ds\right) \\
&-\int_0^t \exp\left(C\int_s^t \|\Sigma(w)\|_{L^\infty}(\tau)d\tau\right)\int_{\mathbf{T}^3} E(w)\cdot(U-w)(s,x)dxds\,.
\end{aligned}
$$

Set

$$
\begin{aligned}
h_\varepsilon[w](t) :&= \frac{1}{\varepsilon^2}\int_{\mathbf{T}^3} \tfrac{1}{2}\rho_{F_\varepsilon}|u_{F_\varepsilon}-\varepsilon w|^2(t,x)dx \\
&= \sup_{b\in C_b(\mathbf{T}^3;\mathbf{R}^3)} \int_{\mathbf{T}^3}\left(\frac{1}{\varepsilon}(u_{F_\varepsilon}-\varepsilon w)\cdot b - \tfrac{1}{2}|b|^2\right)\rho_{F_\varepsilon}dx \\
&= \mathscr{F}\left[\rho_{F_\varepsilon}(t,\cdot), \rho_{F_\varepsilon}\frac{1}{\varepsilon}(u_{F_\varepsilon}-\varepsilon w)(t,\cdot)\right].
\end{aligned}
$$

Observe that $\mathscr{F}$ is a jointly weakly l.s.c. and convex functional on the class of bounded, vector valued Radon measures on $\mathbf{T}^3$. Besides

$$
\begin{aligned}
h_\varepsilon[w](t) :&= \frac{1}{\varepsilon^2}\int_{\mathbf{T}^3} \tfrac{1}{2}\rho_{F_\varepsilon}|u_{F_\varepsilon}-\varepsilon w|^2(t,x)dx \\
&\leq \frac{1}{\varepsilon^2}H(M_{F_\varepsilon}|\mathscr{M}_{(1,\varepsilon w,1)})(t) \leq \frac{1}{\varepsilon^2}H(F_\varepsilon|\mathscr{M}_{(1,\varepsilon w,1)})(t) \leq C^{in}\,.
\end{aligned}
$$

By the Banach-Alaoglu theorem, possibly after extracting subsequences, one has

$$
\rho_{F_\varepsilon}(t,\cdot)\rightharpoonup 1\,, \quad \rho_{F_\varepsilon}\frac{1}{\varepsilon}(u_{F_\varepsilon}-\varepsilon w)(t,\cdot)\rightharpoonup (U-w)(t,\cdot)
$$

in the weak topology of measures on $\mathbf{T}^3$, while

$$
h_\varepsilon[w](t)\rightharpoonup h_w(t)\leq H_w(t)
$$

in $L^\infty([0,T])$ weak-*. Moreover

$$
\begin{aligned}
\mathscr{F}[1,(U-w)(t,\cdot)] \leq h_w(t) &\leq H_w(0)\exp\left(C\int_0^t \|\Sigma(w)\|_{L^\infty}(s)ds\right) \\
&-\int_0^t \exp\left(C\int_s^t \|\Sigma(w)\|_{L^\infty}(\tau)d\tau\right)\int_{\mathbf{T}^3} E(w)\cdot(U-w)(s,x)dxds\,.
\end{aligned}
$$

Observing that

$$
\mathscr{F}[1,(U-w)(t,\cdot)] = \tfrac{1}{2}\int_{\mathbf{T}^3}|U-w|^2(t,x)dx
$$

while

$$
\begin{aligned}
&\frac{1}{\varepsilon^2}H(\mathscr{M}_{(1,\varepsilon u^{in},1)}|\mathscr{M}_{(1,\varepsilon w(0,\cdot),1)}) \\
&\qquad = \tfrac{1}{2}\int_{\mathbf{T}^3}|u^{in}(x)-w(0,x)|^2dx = H_w(0)\,,
\end{aligned}
$$

we conclude that

$$
\begin{aligned}
&\tfrac{1}{2}\int_{\mathbf{T}^3} |U-w|^2(t,x)dx \\
&\le \tfrac{1}{2}\int_{\mathbf{T}^3} |u^{in}(x)-w(0,x)|^2 dx \exp\left(\int_0^t C\|\Sigma(w)\|_{L^\infty}(s)ds\right) \\
&+\int_0^t \exp\left(\int_s^t C\|\Sigma(w)\|_{L^\infty}(\tau)d\tau\right)\int_{\mathbf{T}^3} E(w)\cdot(U-w)(s,x)dxds\,.
\end{aligned}
$$

In other words, U satisfies an inequality analogous to the one defining the notion of dissipative solution — up to replacing the constant C with 2.

By the same argument as the one proving the uniqueness of classical solutions of Euler's equation within the class of dissipative solutions, setting $w=u$ (the solution of the Cauchy problem for the Euler equation with initial data u^{in} defined on $[0,T^*)$ for each $T<T^*$), one has

$$
\begin{aligned}
&\tfrac{1}{2}\int_{\mathbf{T}^3} |U-u|^2(t,x)dx \\
&\le \int_0^t \exp\int_s^t C\|\Sigma(w)\|_{L^\infty}(\tau)d\tau\int_{\mathbf{T}^3} E(u)\cdot(U-u)(s,x)dxds=0\,,
\end{aligned}
$$

since

$$
\begin{aligned}
\int_{\mathbf{T}^3} E(u)\cdot(U-u)(s,x)dx &= \int_{\mathbf{T}^3} -\nabla_x p\cdot(U-u)(s,x)dx \\
&= \int_{\mathbf{T}^3} p\,\mathrm{div}_x(U-u)(s,x)dx=0\,.
\end{aligned}
$$

This completes the proof of Theorem 2.7.

3 Lecture 3: The Incompressible Navier-Stokes Limit

The incompressible Navier-Stokes limit is the only nonlinear regime where the fluid dynamic limit of the Boltzmann equation is known to hold without any restriction on the time interval on which the limit is valid, or on the size and regularity of the initial distribution function. It connects two analogous theories of global weak solutions, the Leray existence theory of weak solutions of the incompressible Navier-Stokes equation, and the DiPerna-Lions theory of renormalized solutions of the Boltzmann equation. This last lecture will give an idea of the proof of the fluid dynamic limit in this regime.

For the sake of simplicity, we consider only the Navier-Stokes motion equation, without the drift-diffusion equation for the temperature. In other words, this lecture is focussed on the following theorem, that is a slightly simpler variant of the incompressible Navier-Stokes limit theorem presented in lecture 1.

Theorem 3.1 (F. Golse-L. Saint-Raymond [38, 40]) *Let F_ε be a family of renormalized solutions of the Cauchy problem for the Boltzmann equation with initial data*

$$F_\varepsilon\big|_{t=0} = \mathscr{M}_{(1,\varepsilon u^{in}(\varepsilon x),1)}\,,$$

where $u^{in} \in L^2(\mathbf{R}^3)$ *satisfies* $\operatorname{div}_x u^{in} = 0$*. For some subsequence* $\varepsilon_n \to 0$*, one has*

$$\frac{1}{\varepsilon_n}\int_{\mathbf{R}^3} v(F_{\varepsilon_n}\left(\frac{t}{\varepsilon_n^2},\frac{x}{\varepsilon_n},v\right)dv \to u(t,x) \text{ weakly in } L^1_{loc}\,,$$

where u *is a Leray solution with initial data* u^{in} *of*

$$\partial_t u + \operatorname{div}_x(u\otimes u) + \nabla_x p = \nu\Delta_x u\,, \quad \operatorname{div}_x u = 0\,.$$

The viscosity ν *is given by the same formula as in (8), recalled below:*

$$\nu = \tfrac{1}{5}\mathscr{D}^*(v\otimes v - \tfrac{1}{3}|v|^2 I)\,,$$

where $\mathscr{D}$ *is the quadratic functional*

$$\mathscr{D}(\Phi) = \tfrac{1}{8}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}|\Phi+\Phi_*-\Phi'-\Phi'_*|^2|(v-v_*)\cdot\omega|MM_*dvdv_*d\omega\,,$$

and $\mathscr{D}^*$ *its Legendre dual.*

We also recall that a Leray solution of the incompressible Navier-Stokes equation is a divergence free vector field

$$u \in C(\mathbf{R}_+; w-L^2(\mathbf{R}^3)) \cap L^2(\mathbf{R}_+; H^1(\mathbf{R}^3))$$

such that

$$\frac{d}{dt}\int_{\mathbf{R}^3} u(t,x)\cdot w(x)dx + \nu\int_{\mathbf{R}^3}\nabla_x u(t,x) : \nabla w(x)dx = \int_{\mathbf{R}^3}\nabla w(x) : u(t,x)\otimes u(t,x)dx$$

in the sense of distributions on $\mathbf{R}_+^*$ for each divergence free vector field w in the Sobolev space $H^1(\mathbf{R}^3)$, together with the energy inequality

$$\tfrac{1}{2}\int_{\mathbf{R}^3}|u(t,x)|^2dx + \nu\int_{\mathbf{R}^3}|\nabla_x u|^2dx \le \tfrac{1}{2}\int_{\mathbf{R}^3}|u(0,x)|^2dx$$

for all $t \ge 0$. The reader is referred to the original work of J. Leray [52] for more details on this notion, together with [20] or chapter 3 in [55].

3.1 Formal Derivation of the Incompressible Navier-Stokes Equations from the Boltzmann Equation

3.1.1 The Rescaled Boltzmann Equation

The incompressible Navier-Stokes scaling for the Boltzmann equation assumes that the Knudsen, Mach and Strouhal numbers satisfy $\mathrm{Kn} = \mathrm{Ma} = \mathrm{Sh} = \varepsilon$ (in the terminology introduced at the end of the Lecture 1) so that $\mathrm{Re} = 1$ (by the von Karman relation).

In other words, the assumption $\mathrm{Kn} = \mathrm{Sh} = \varepsilon$ means that, if F is the distribution function that is the solution of the Boltzmann equation, the incompressible Navier-Stokes limit involves the rescaled distribution function $F_\varepsilon(t,x,v) := F(t/\varepsilon^2, x/\varepsilon, v)$. This rescaled distribution function satisfies the rescaled Boltzmann equation

$$\varepsilon\partial_t F_\varepsilon + v\cdot\nabla_x F_\varepsilon = \frac{1}{\varepsilon}\mathscr{C}(F_\varepsilon)\,.$$

On the other hand, the assumption $\mathrm{Ma} = \varepsilon$ indicates that F_ε is sought as an $O(\varepsilon)$ perturbation of the uniform Maxwellian equilibrium $M := \mathscr{M}_{(1,0,1)}$, i.e. that one has

$$F_\varepsilon(t,x,v) = M(v)G_\varepsilon(t,x,v)\,,\quad G_\varepsilon(t,x,v) = 1+\varepsilon g_\varepsilon(t,x,v)\,,$$

with $g_\varepsilon = O(1)$ as $\varepsilon \to 0$.

The proof of the incompressible Navier-Stokes limit of the Boltzmann equation that we discuss below is not based on Hilbert's expansion — as explained in Lecture 1, Hilbert's expansion truncated as in [17, 23] may fail to guarantee the positivity of the distribution function, and may break down if the solution of the Navier-Stokes equations lose regularity in finite time — a problem still open in the 3-dimensional case at the time of this writing.

For that reason, a more robust moment method was proposed by Bardos-Golse-Levermore in [7]. This method leads to a formal argument for the incompressible Navier-Stokes limit that is very close to the structure of the complete proof. For that reason, we first present this formal argument before sketching the proof itself.

In terms of the relative number density fluctuation g_ε, the scaled Boltzmann equation becomes

$$\varepsilon\partial_t g_\varepsilon + v\cdot\nabla_x g_\varepsilon + \frac{1}{\varepsilon}\mathscr{L}g_\varepsilon = \mathscr{Q}(g_\varepsilon, g_\varepsilon)\,.$$

This form of the rescaled Boltzmann equation involves the linearized collision integral (intertwined with M), denoted

$$\mathscr{L}g := -M^{-1}D\mathscr{C}(M)\cdot(Mg)\,,$$

together with the Hessian of the collision integral (intertwined with M), denoted

$$\mathscr{Q}(g,g) := \tfrac{1}{2}M^{-1}\mathscr{C}(Mg)\,.$$

3.1.2 The Linearized Collision Integral

The explicit form of $\mathscr{L}$ is as follows:

$$\mathscr{L}g(v) := \iint_{\mathbf{R}^3\times\mathbf{S}^2}(g(v)+g(v_*)-g(v')-g(v'_*))|(v-v_*)\cdot\omega|M(v_*)dv_*d\omega\,.$$

Theorem 3.2 (D. Hilbert [47]) *The linearized collision integral operator $\mathscr{L}$ is a self-adjoint, nonnegative, Fredholm, unbounded operator on $L^2(\mathbf{R}^3;Mdv)$ with domain*

$$\mathrm{Dom}\,\mathscr{L} = L^2(\mathbf{R}^3;(1+|v|)Mdv)$$

and nullspace

$$\mathrm{Ker}\,\mathscr{L} = \mathrm{span}\{1,v_1,v_2,v_3,|v|^2\}\,.$$

3.1.3 Asymptotic Fluctuations

Multiplying the Boltzmann equation by ε and letting $\varepsilon\to 0$ suggests that

$$g_\varepsilon \to g \text{ as } \varepsilon\to 0\,,\quad \text{ with } \mathscr{L}g=0\,.$$

By Hilbert's theorem, g is an *infinitesimal Maxwellian*, meaning that $g(t,x,v)$ is of the form

$$g(t,x,v) = \rho(t,x)+u(t,x)\cdot v+\tfrac{1}{2}\theta(t,x)(|v|^2-3)\,.$$

Notice that, in this case, g is parametrized by its own moments in the v variable, since

$$\rho = \langle g\rangle\,,\quad u=\langle vg\rangle\,,\quad \text{ and } \theta = \langle(\tfrac{1}{3}|v|^2-1)g\rangle\,.$$

This observation is important in the rigorous derivation of the incompressible Navier-Stokes equations from the Boltzmann equation.

Henceforth, we systematically use the following notation.

Notation: for all $\phi\in L^1(\mathbf{R}^3;Mdv)$, one denotes

$$\langle\phi\rangle := \int_{\mathbf{R}^3}\phi(v)M(v)dv\,.$$

3.1.4 The Incompressibility and Boussinesq Relations

The continuity equation (local conservation of mass) reads

$$\varepsilon\partial_t\langle g_\varepsilon\rangle + \mathrm{div}_x\langle vg_\varepsilon\rangle = 0\,,$$

and passing to the limit in the sense of distributions, we expect that

$$\langle vg_\varepsilon\rangle \to \langle vg\rangle = u\,,\quad \text{ and thus } \mathrm{div}_x\langle vg\rangle = \mathrm{div}_x u = 0\,.$$

This is incompressibility condition in the Navier-Stokes equations.

Likewise, the local conservation of momentum takes the form

$$\varepsilon\partial_t\langle vg_\varepsilon\rangle+\operatorname{div}_x\langle v\otimes vg_\varepsilon\rangle=0\,.$$

Passing to the limit in the sense of distributions on both sides of the equality above, we expect that

$$\langle v\otimes vg_\varepsilon\rangle\to\langle v\otimes vg\rangle=(\rho+\theta)I\,,$$

(where the last equality follows from straightforward computations) so that

$$\operatorname{div}_x((\rho+\theta)I)=\nabla_x(\rho+\theta)=0\,.$$

The following slight variant of this argument provides insight into the next step of this proof, namely the derivation of the Navier-Stokes motion equation.

Recall that the incompressible Navier-Stokes motion equation is

$$\partial_t u+u\cdot\nabla_x u-\nu\Delta_x u=-\nabla_x p\,,$$

and that it involves the term $\nabla_x p$ as the Lagrange multiplier associated to the constraint $\operatorname{div}_x u=0$. Accordingly, we split the tensor $v\otimes v$ into its traceless and scalar component:

$$v\otimes v=\left(v\otimes v-\tfrac{1}{3}|v|^2I\right)+\tfrac{1}{3}|v|^2I\,,$$

so that the local conservation of momentum becomes

$$\varepsilon\partial_t\langle vg_\varepsilon\rangle+\operatorname{div}_x\langle Ag_\varepsilon\rangle+\nabla_x\langle\tfrac{1}{3}|v|^2g_\varepsilon\rangle=0\,,$$

where

$$A(v)=v\otimes v-\tfrac{1}{3}|v|^2I\,.$$

The key observation is that

$$A\perp\operatorname{Ker}\mathscr{L}\,;$$

see Appendix 2 (and especially Lemma 5.3.

Passing to the limit in the local conservation of momentum above in the sense of distributions, we expect that

$$\langle Ag_\varepsilon\rangle\to\langle Ag\rangle=0\text{ since }g(t,x,\cdot)\in\operatorname{Ker}\mathscr{L}\text{ for a.e. }(t,x)\in\mathbf{R}_+\times\mathbf{R}^3\,.$$

On the other hand

$$\langle\tfrac{1}{3}|v|^2g_\varepsilon\rangle\to\langle\tfrac{1}{3}|v|^2g\rangle=\rho+\theta\,.$$

Thus

$$\operatorname{div}_x\langle Ag\rangle+\nabla_x\langle\tfrac{1}{3}|v|^2g\rangle=\nabla_x(\rho+\theta)=0\,.$$

If $g\in L^\infty(\mathbf{R}_+;L^2(\mathbf{R}^3;Mdvdx))$, this implies the Boussinesq relation

$$\rho+\theta=0\,,\qquad\text{so that }g(t,x,v)=u(t,x)\cdot v+\theta(t,x)\tfrac{1}{2}(|v|^2-5)\,.$$

3.1.5 The Motion Equation

It remains to derive the Navier-Stokes motion equation. Start from the local conservation of momentum in the form

$$\partial_t \langle v g_\varepsilon \rangle + \mathrm{div}_x \frac{1}{\varepsilon} \langle A g_\varepsilon \rangle + \nabla_x \frac{1}{\varepsilon} \langle \tfrac{1}{3} |v|^2 g_\varepsilon \rangle = 0$$

As mentionned above, $A_{kl} \perp \mathrm{Ker}\mathscr{L}$ for all $k, l = 1, 2, 3$. Applying the Fredholm alternative to the linearized collision integral $\mathscr{L}$ shows the existence of a unique tensor field $\hat{A} \in \mathrm{Dom}(\mathscr{L})$ such that

$$A_{kl} = \mathscr{L}\hat{A}_{kl}\,, \quad \text{ and } \hat{A}_{kl} \perp \mathrm{Ker}\mathscr{L} \text{ for all } k, l = 1, 2, 3\,.$$

Therefore

$$\begin{aligned} \frac{1}{\varepsilon}\langle A g_\varepsilon \rangle = \left\langle \hat{A} \frac{1}{\varepsilon} \mathscr{L} g_\varepsilon \right\rangle &= \langle \hat{A} \mathscr{Q}(g_\varepsilon, g_\varepsilon) \rangle - \langle \hat{A}(\varepsilon \partial_t + v \cdot \nabla_x) g_\varepsilon \rangle \\ &\to \langle \hat{A} \mathscr{Q}(g, g) \rangle - \langle \hat{A} v \cdot \nabla_x g \rangle \end{aligned}$$

as $\varepsilon \to 0$.

Since g is an infinitesimal Maxwellian and ρ, θ satisfy the Boussinesq relation, one has

$$g = u \cdot v + \theta \tfrac{1}{2}(|v|^2 - 5)\,,$$

so that

$$\begin{aligned} \langle \hat{A} v \cdot \nabla_x g \rangle &= \tfrac{1}{2} \langle \hat{A} \otimes A \rangle : D(u) + \langle \hat{A} \otimes \tfrac{1}{2}(|v|^2 - 5) v \rangle \cdot \nabla_x \theta \\ &= \tfrac{1}{2} \langle \hat{A} \otimes A \rangle : D(u) \text{ since } \hat{A} \text{ is even,} \end{aligned}$$

where

$$D(u) := \nabla_x u + (\nabla_x u)^T - \tfrac{2}{3} \mathrm{div}_x u I$$

is the traceless deformation tensor of u. Notice that $\langle \hat{A} |v|^2 \rangle = 0$ since $\hat{A}_{kl} \perp \mathrm{Ker}\mathscr{L}$ for all $k, l = 1, 2, 3$, so that

$$\langle \hat{A} \otimes (v \otimes v) \rangle = \langle \hat{A} \otimes A \rangle\,.$$

It remains to compute the term $\langle \hat{A} \mathscr{Q}(g, g) \rangle$. This is done with the next lemma.

Lemma 3.3 (C. Cercignani [18], C. Bardos-F. Golse-C.D. Levermore [8]) *Each infinitesimal Maxwellian $g \in \mathrm{Ker}\mathscr{L}$ satisfies the relation*

$$\mathscr{Q}(g, g) = \tfrac{1}{2} \mathscr{L}(g^2)\,.$$

Proof. Differentiate twice the relation $\mathscr{C}(\mathscr{M}_{(\rho,u,\theta)}) = 0$, and observe that the range of the differential $d\mathscr{M}_{(\rho,u,\theta)}$ is equal to $\mathrm{Ker}\mathscr{L}$.

With this observation, one has

$$\langle \hat{A}\mathscr{Q}(g,g)\rangle = \tfrac{1}{2}\langle \hat{A}\mathscr{L}(g^2)\rangle = \tfrac{1}{2}\langle Ag^2\rangle = \tfrac{1}{2}\langle A\otimes A\rangle : \left(u\otimes u - \tfrac{1}{3}|u|^2 I\right).$$

Therefore

$$\frac{1}{\varepsilon}\langle Ag_\varepsilon\rangle \to \tfrac{1}{2}\langle A\otimes A\rangle : \left(u\otimes u - \tfrac{1}{3}|u|^2 I\right) - \tfrac{1}{2}\langle \hat{A}\otimes A\rangle : D(u).$$

Lemma 3.4 *For all $i,j,k,l\in\{1,2,3\}$, one has*

$$\langle A_{ij}A_{kl}\rangle = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}\,,$$
$$\langle \hat{A}_{ij}A_{kl}\rangle = \nu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}\right),$$

where

$$\nu = \tfrac{1}{10}\langle \hat{A} : A\rangle > 0$$

is the viscosity.

The proof of this Lemma will be given in Appendix 2.

Thus

$$\frac{1}{\varepsilon}\langle Ag_\varepsilon\rangle \to \left(u\otimes u - \tfrac{1}{3}|u|^2 I\right) - \nu D(u).$$

Substituting this expression in the momentum conservation laws shows that

$$\partial_t u + \operatorname{div}_x(u\otimes u) - \nu \operatorname{div}_x D(u) + \operatorname{div}_x(\tfrac{1}{3}|u|^2 I) + \nabla_x \frac{1}{\varepsilon}\langle \tfrac{1}{3}|v|^2 g_\varepsilon\rangle = 0,$$

or equivalently

$$\partial_t u + \operatorname{div}_x(u\otimes u) - \nu\Delta_x u = -\nabla_x\left(\frac{1}{\varepsilon}\langle \tfrac{1}{3}|v|^2 g_\varepsilon\rangle - \tfrac{1}{3}|u|^2\right).$$

Indeed, the divergence free condition $\operatorname{div}_x u = 0$ implies that

$$\operatorname{div}_x D(u) = \Delta_x u + \nabla_x(\operatorname{div}_x u) - \tfrac{2}{3}\nabla_x(\operatorname{div}_x u) = \Delta_x u.$$

Equivalently

$$\partial_t u + \operatorname{div}_x(u\otimes u) - \nu\Delta_x u = 0 \text{ modulo gradient fields.}$$

Let the Dirichlet form for the linearized collision integral $\mathscr{L}$ be defined as follows:

$$\mathscr{D}(\Phi) := \tfrac{1}{2}\langle \Phi : \mathscr{L}\Phi\rangle.$$

As explained in lecture 1, the formula for the viscosity can be put in the form

$$\nu = \tfrac{1}{5}\mathscr{D}^*(A),$$

where $\mathscr{D}^*$ designates the Legendre dual of $\mathscr{D}$. Indeed, since $\mathscr{D}$ is a quadratic functional defined on $\operatorname{Dom}\mathscr{L}\otimes M_3(\mathbf{R}) \simeq (\operatorname{Dom}\mathscr{L})^9$, one has

$$\mathscr{D}^*(\Phi) = \tfrac{1}{2}\langle \Phi : \mathscr{L}^{-1}\Phi \rangle$$

for all $\Phi \in (\mathrm{Ker}\,\mathscr{L})^\perp$. Applying this to $\Phi = \hat{A}$ gives back the formula in Lemm 3.4.

3.2 Sketch of the Proof of the Incompressible Navier-Stokes Limit of the Boltzmann Equation

The complete proof of the incompressible Navier-Stokes limit of the Boltzmann equation is quite involved (see [38, 40]). Therefore we only sketch the main steps in the argument.

3.2.1 The Strategy

First we choose a convenient normalizing nonlinearity for the Boltzmann equation. Pick $\gamma \in C^\infty(\mathbf{R}_+)$, a nonincreasing function such that

$$\gamma\big|_{[0,3/2]} \equiv 1\,, \quad \gamma\big|_{[2,+\infty)} \equiv 0\,; \qquad \text{and set } \hat{\gamma}(z) = \frac{d}{dz}((z-1)\gamma(z))\,.$$

The Boltzmann equation is renormalized relatively to M as follows

$$\partial_t(g_\varepsilon\gamma_\varepsilon) + \frac{1}{\varepsilon}v\cdot\nabla_x(g_\varepsilon\gamma_\varepsilon) = \frac{1}{\varepsilon^3}\hat{\gamma}_\varepsilon\mathscr{Q}(G_\varepsilon,G_\varepsilon)\,,$$

where

$$\gamma_\varepsilon := \gamma(G_\varepsilon) \quad \text{ while } \hat{\gamma}_\varepsilon := \hat{\gamma}(G_\varepsilon)\,.$$

We recall the notation $\mathscr{Q}(G,G) = M^{-1}\mathscr{C}(MG)$.

Renormalized solutions of the Boltzmann equation satisfy the local conservation law of mass:

$$\varepsilon\partial_t\langle g_\varepsilon\rangle + \mathrm{div}_x\langle vg_\varepsilon\rangle = 0\,.$$

The entropy bound and Young's inequality imply that

$$(1+|v|^2)g_\varepsilon \text{ is relatively compact in } w-L^1_{loc}(dtdx;L^1(Mdv))\,.$$

Therefore, modulo extraction of a subsequence,

$$g_\varepsilon \to g \text{ weakly in } L^1_{loc}(dtdx;L^1(M(1+|v|^2)dv))\,.$$

Hence

$$\langle vg_\varepsilon\rangle \to \langle vg\rangle =: u \quad \text{ weakly in } L^1_{loc}(\mathbf{R}_+\times\mathbf{R}^3)\,.$$

Passing to the limit in the continuity equation leads to the incompressibility condition:

$$\operatorname{div}_x u = 0 .$$

Since high velocities are a source of difficulties in the hydrodynamic limit, we shall use a special truncation procedure, defined as follows. Pick $K > 6$ and set $K_\varepsilon = K|\ln\varepsilon|$; for each function $\xi \equiv \xi(v)$, define

$$\xi_{K_\varepsilon}(v) := \xi(v)\mathbf{1}_{|v|^2 \le K_\varepsilon} .$$

Multiply both sides of the scaled, renormalized Boltzmann equation by each component of v_{K_ε}: one finds that

$$\partial_t \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle + \operatorname{div}_x \mathbf{F}_\varepsilon(A) + \nabla_x \frac{1}{\varepsilon} \langle \tfrac{1}{3}|v|^2_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle = \mathbf{D}_\varepsilon(v) ,$$

where

$$\begin{cases} \mathbf{F}_\varepsilon(A) := \frac{1}{\varepsilon}\langle A_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle , \\ \mathbf{D}_\varepsilon(v) := \dfrac{1}{\varepsilon^3} \left\langle\!\left\langle v_{K_\varepsilon} \hat{\gamma}_\varepsilon (G'_\varepsilon G'_{\varepsilon *} - G_\varepsilon G_{\varepsilon *}) \right\rangle\!\right\rangle . \end{cases}$$

We recall that

$$\langle \phi \rangle = \int_{\mathbf{R}^3} \phi(v) M dv ,$$

and introduce a new element of notation

$$\langle\!\langle \psi \rangle\!\rangle := \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} \psi(v, v_*, \omega) d\mu ,$$

where

$$d\mu := |(v - v_*)\cdot\omega| M dv M_* dv_* d\omega .$$

With the notation introduced above, our goal is to prove that, modulo extraction of a subsequence,

$$\begin{aligned} &\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \to \langle vg \rangle =: u && \text{weakly in } L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3) , \\ &\mathbf{D}_\varepsilon(v) \to 0 && \text{strongly in } L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^3) , \text{ and} \\ &P(\operatorname{div}_x \mathbf{F}_\varepsilon(A)) \to P \operatorname{div}_x(u^{\otimes 2}) - \nu \Delta_x u && \text{weakly in } L^1_{loc}(\mathbf{R}_+, W^{-s,1}_{loc}(\mathbf{R}^3)) , \end{aligned}$$

for $s > 1$ as $\varepsilon \to 0$, where P denotes the Leray projection, i.e. the orthogonal projection on divergence-free vector fields in $L^2(\mathbf{R}^3)$.

See section 2.4 in [40] for the missing details.

3.2.2 Uniform A Priori Estimates

The only uniform a priori estimate satisfied by renormalized solutions of the Boltzmann equation comes from the DiPerna-Lions entropy inequality:

$$H(F_\varepsilon|M)(t)+\frac{1}{\varepsilon^2}\int_0^t\int_{\mathbf{R}^3}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} d(F_\varepsilon)|(v-v_*)\cdot\omega|dvdv_*d\omega dxds$$
$$\leq H(F_\varepsilon^{in}|M)=\tfrac{1}{2}\varepsilon^2\|u^{in}\|_{L^2}^2\,,$$

where the entropy production integrand is denoted

$$d(f):=\tfrac{1}{4}(f'f'_*-ff_*)\ln\left(\frac{f'f'_*}{ff_*}\right).$$

We also recall the following elementary, pointwise inequalities:

$$(\sqrt{Z}-1)^2\leq Z\ln Z-Z+1\,,\quad 4(\sqrt{X}-\sqrt{Y})^2\leq(X-Y)\ln(X/Y)\,,$$

for all $X,Y,Z>0$.

With the DiPerna-Lions entropy inequality, and the pointwise inequalities above, one gets the following bounds that are uniform in ε:

$$\int_{\mathbf{R}^3}\langle(\sqrt{G_\varepsilon}-1)^2\rangle dx\leq C\varepsilon^2\,,$$
$$\int_0^{+\infty}\int_{\mathbf{R}^3}\left\langle\!\!\left\langle\left(\sqrt{G'_\varepsilon G'_{\varepsilon*}}-\sqrt{G_\varepsilon G_{\varepsilon*}}\right)^2\right\rangle\!\!\right\rangle dxdt\leq C\varepsilon^4\,.$$

This is precisely Proposition 2.3 in [40].

3.2.3 Vanishing of Conservation Defects

Since renormalized solutions of the Boltzmann equation are not known to satisfy the local conservation laws of momentum and energy, one has to consider instead the local conservation laws of moments of renormalized distribution functions, truncated at high velocities, modulo conservation defects. The idea is to prove that the conservation defects vanish in the hydrodynamic limit. In other words, even if the local conservation of momentum and energy are not known to be satisfied by renormalized solutions of the Boltzmann equation, they are satisfied in the hydrodynamic limit.

This approach was proposed for the first time in [10]. The procedure for proving the vanishing of conservation defects was formulated in essentially the most general possible setting can be found in [32], and applied to the acoustic and Stokes-Fourier limits. The statement below is taken from [40], it is more general and slightly less technical than the analogous result in [38].

Proposition 3.5 *The conservation defect*

$$\mathbf{D}_\varepsilon(v):=\frac{1}{\varepsilon^3}\left\langle\!\!\left\langle v_{K_\varepsilon}\hat{\gamma}_\varepsilon(G'_\varepsilon G'_{\varepsilon*}-G_\varepsilon G_{\varepsilon*})\right\rangle\!\!\right\rangle$$

satisfies

$$\mathbf{D}_\varepsilon(v)\to 0$$

in $L^1_{loc}(\mathbf{R}_+\times\mathbf{R}^3)$ *as* $\varepsilon\to 0$.

This is Proposition 5.1 in [40].

Proof. Split the conservation defect as $\mathbf{D}_\varepsilon(v)=\mathbf{D}^1_\varepsilon(v)+\mathbf{D}^2_\varepsilon(v)$ with

$$\mathbf{D}^1_\varepsilon(v):=\frac{1}{\varepsilon^3}\left\langle\!\left\langle v_{K_\varepsilon}\hat{\gamma}_\varepsilon\left(\sqrt{G'_\varepsilon G'_{\varepsilon*}}-\sqrt{G_\varepsilon G_\varepsilon}\right)^2\right\rangle\!\right\rangle,$$
$$\mathbf{D}^2_\varepsilon(v):=\frac{2}{\varepsilon^3}\left\langle\!\left\langle v_{K_\varepsilon}\hat{\gamma}_\varepsilon\left(\sqrt{G'_\varepsilon G'_{\varepsilon*}}-\sqrt{G_\varepsilon G_\varepsilon}\right)\sqrt{G_\varepsilon G_\varepsilon}\right\rangle\!\right\rangle.$$

That $\mathbf{D}^1_\varepsilon(v)\to 0$ follows from the entropy production estimate.

Setting

$$\Xi_\varepsilon:=\frac{1}{\varepsilon^2}\left(\sqrt{G'_\varepsilon G'_{\varepsilon*}}-\sqrt{G_\varepsilon G_\varepsilon}\right)\sqrt{G_\varepsilon G_\varepsilon},$$

we split $\mathbf{D}^2_\varepsilon(v)$ as

$$\begin{aligned}\mathbf{D}^2_\varepsilon(v)=&-\frac{2}{\varepsilon}\left\langle\!\left\langle v\mathbf{1}_{|v|^2>K_\varepsilon}\hat{\gamma}_\varepsilon\Xi_\varepsilon\right\rangle\!\right\rangle+\frac{2}{\varepsilon}\left\langle\!\left\langle v\hat{\gamma}_\varepsilon(1-\hat{\gamma}_{\varepsilon*}\hat{\gamma}'_\varepsilon\hat{\gamma}_{\varepsilon*})\Xi_\varepsilon\right\rangle\!\right\rangle\\&+\frac{1}{\varepsilon}\left\langle\!\left\langle(v+v_1)\hat{\gamma}_\varepsilon\hat{\gamma}_{\varepsilon*}\hat{\gamma}'_\varepsilon\hat{\gamma}_{\varepsilon*}\Xi_\varepsilon\right\rangle\!\right\rangle.\end{aligned}$$

The first and third terms are mastered by the entropy production bound and classical estimates on the tail of Gaussian distributions. See Lemma 5.2 in [40] and the discussion on pp. 530–531.

Sending the second term to 0 requires knowing that

$$(1+|v|)\left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2\text{ is uniformly integrable on }[0,T]\times K\times\mathbf{R}^3$$

for the measure $dtdxMdv$, for each $T>0$ and each compact $K\subset\mathbf{R}^3$. See [40] on pp. 531–532 for the (rather involved) missing details.

3.2.4 Asymptotic Behavior of the Momentum Flux

We recall that the momentum flux is defined by the formula

$$\mathbf{F}_\varepsilon(A)=\tfrac{1}{\varepsilon}\langle A_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle.$$

Proposition 3.6 *Denoting by* Π *the* $L^2(Mdv)$*-orthogonal projection on* $\operatorname{Ker}\mathscr{L}$*, one has*

$$\mathbf{F}_\varepsilon(A)=2\left\langle A\left(\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2\right\rangle-2\left\langle\hat{A}\frac{1}{\varepsilon^2}\mathscr{Q}(\sqrt{G_\varepsilon},\sqrt{G_\varepsilon})\right\rangle+o(1)_{L^1_{loc}(dtdx)}.$$

This is Proposition 6.1 in [40].

The proof is based upon splitting $\mathbf{F}_\varepsilon(A)$ as

$$\mathbf{F}_\varepsilon(A) = \left\langle A_{K_\varepsilon}\gamma_\varepsilon \left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2\right\rangle + \frac{2}{\varepsilon}\left\langle A_{K_\varepsilon}\gamma_\varepsilon \frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right\rangle,$$

by uniform integrability of $(1+|v|)\left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2$, implying in turn that

$$\lim_{\varepsilon\to 0}\left\|\frac{\sqrt{G_\varepsilon}-1}{\varepsilon} - \Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right\|_{L^2_{loc}(dtdx;L^2((1+|v|)Mdv))} = 0\,.$$

By the entropy production bound, up to extraction of a subsequence

$$\frac{1}{\varepsilon^2}\left(\sqrt{G'_\varepsilon G'_{\varepsilon *}} - \sqrt{G_\varepsilon G_\varepsilon}\right) \to q \text{ weakly in } L^2(dtdxd\mu)\,.$$

Passing to the limit in the scaled, renormalized Boltzmann equation:

$$\iint_{\mathbf{R}^3\times\mathbf{S}^2} q|(v-v_*)\cdot\omega|M_* dv_* d\omega = \tfrac{1}{2}v\cdot\nabla_x g = \tfrac{1}{2}A:\nabla_x u + \text{ odd function of } v\,.$$

Since $\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\simeq\frac{1}{2}g_\varepsilon\gamma_\varepsilon$, one gets

$$\mathbf{F}_\varepsilon(A) = A(\langle v_{K_\varepsilon} g_\varepsilon\gamma_\varepsilon\rangle) - \nu(\nabla_x u + (\nabla_x u)^T) + o(1)_{w-L^1_{loc}(dtdx)}\,,$$

(we recall the notation $A(u) := u\otimes u - \frac{1}{3}|u|^2 I$), while

$$\langle v_{K_\varepsilon} g_\varepsilon\gamma_\varepsilon\rangle \to u \text{ weakly in } L^1_{loc}(\mathbf{R}_+\times\mathbf{R}^3)\,.$$

3.3 Strong Compactness

Because the Navier-Stokes equation is nonlinear, weak compactness of truncated variants of the relative fluctuations of the distribution functions is not enough to prove the fluid dynamic limit. Proving that some appropriate quantities, such as $\langle v_{K_\varepsilon} g_\varepsilon\gamma_\varepsilon\rangle$, defined in terms of renormalized solutions of the Boltzmann equation are relatively compact in the strong topology of L^2 is an essential step in order to pass to the limit in the quadratic term $A(\langle v_{K_\varepsilon} g_\varepsilon\gamma_\varepsilon\rangle)$.

For that purpose, we appeal to “velocity averaging” theorems, a special class of regularity/compactness results on velocity averages of solutions of kinetic equations — see [1, 35, 34, 26].

Before discussing these results in detail, we recall the following elementary observations.

It is well known that, if $F\equiv F(x)$ and $R\equiv R(x)$ satisfy both $F,R\in L^2(\mathbf{R}^N)$ and $\Delta F = R$, then F belongs to the Sobolev space $H^2(\mathbf{R}^N)$ — in other words, knowing that

$$F \text{ and } \sum_{i=1}^{N} \partial_{x_i}^2 F \in L^2(\mathbf{R}^N) \text{ implies that } \partial_{x_i}\partial_{x_j} F \in L^2(\mathbf{R}^N) \text{ for } i,j = 1,\ldots,N\,.$$

The analogous question with the advection operator in the place of the Laplacian is as follows: given G and $S \in L^p(\mathbf{R}^N \times \mathbf{R}^N)$ such that $v \cdot \nabla_x G = S$, what is the regularity of G in the x-variable? For instance, does this imply that the function $G \in L^p(\mathbf{R}^N_v; W^{1,p}(\mathbf{R}^N_x))$?

This question is answered in the negative.

For instance, in space dimension $N = 2$, take $\gamma = \mathbf{1}_A$ with A measurable and bounded, and set $G(x,v) = \gamma(x_1 v_2 - x_2 v_1)\mathbf{1}_{|v|\le 1}$. Obviously the function G satisfies $v \cdot \nabla_x G = 0$ and $G \in L^\infty(\mathbf{R}^2 \times \mathbf{R}^2)$ so that $G \in L^p_{loc}(\mathbf{R}^N \times \mathbf{R}^N)$. Yet G does not belong to $W^{s,p}(\mathbf{R}^2)$ for a.e. $v \in \mathbf{R}^2$.

Of course, the reason for the difference between both situations is explained by the fact that the Laplacian is an elliptic operator, while the advection operator is hyperbolic.

3.3.1 Velocity Averaging

The counterexample above suggests that the regularity of G is not the interesting issue to be discussed in the first place. Instead of considering the regularity of G itself, one should instead study the regularity of *velocity averages* of G, i.e. of quantities of the form

$$\int_{\mathbf{R}^3} G(x,v)\phi(v)dv$$

with smooth and compactly supported test function ϕ.

The first result in this direction is the following theorem (see also [1, 35]).

Theorem 3.7 ((F. Golse-P.-L. Lions-B. Perthame-R. Sentis [34]) *Assume that G and S both belong to $L^2(\mathbf{R}^N_x \times \mathbf{R}^N_v)$ and that $v \cdot \nabla_x G = S$. Then, for each $\phi \in C_c(\mathbf{R}^N)$, the velocity average*

$$\mathscr{A}_\phi[G] : x \mapsto \int_{\mathbf{R}^N} G(x,v)\phi(v)dv$$

satisfies $\mathscr{A}_\phi[G] \in H^{1/2}(\mathbf{R}^N)$, with a bound of the form

$$\|\mathscr{A}_\phi[G]\|_{\dot{H}^{1/2}(\mathbf{R}^N_x)} \le C\|G\|^{1/2}_{L^2(\mathbf{R}^N \times \mathbf{R}^N)} \|v \cdot \nabla_x G\|^{1/2}_{L^2(\mathbf{R}^N \times \mathbf{R}^N)}\,.$$

In this statement, the notation $\|\cdot\|_{\dot{H}^s}$ designates the homogeneous H^s seminorm:

$$\|f\|_{\dot{H}^{1/2}(\mathbf{R}^N)} := \left(\iint_{\mathbf{R}^N \times \mathbf{R}^N} \frac{|f(x)-f(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{1/2}\,.$$

In the context of the incompressible Navier-Stokes limit of the Boltzmann equation, the situation is slightly different from the one in the theorem above. Specifically, one has the following controls:

$$\left(\frac{\sqrt{\varepsilon^\alpha+G_\varepsilon}-1}{\varepsilon}\right)^2 \text{ is locally uniformly integrable on } \mathbf{R}_+\times\mathbf{R}^3\times\mathbf{R}^3\,,$$

$$(\varepsilon\partial_t+v\cdot\nabla_x)\frac{\sqrt{\varepsilon^\alpha+G_\varepsilon}-1}{\varepsilon} \text{ is bounded in } L^1_{loc}(\mathbf{R}_+\times\mathbf{R}^3\times\mathbf{R}^3)\,.$$

Mimicking the proof of the velocity averaging theorem above, one deduces from these assumptions that, for each $T>0$ and each compact $C\subset\mathbf{R}^3$,

$$\begin{aligned}\int_0^T\int_C|\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle(t,x+y)-\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle(t,x)|^2dxdt\to 0&\\ \text{as } |y|\to 0\,,\ \text{uniformly in } \varepsilon>0\,.&\end{aligned}\tag{11}$$

See section 4 in [40], especially Proposition 4.4.

3.3.2 Filtering Acoustic Waves

It remains to get compactness in the time variable. Observe that

$$\partial_t P\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle = P(\mathbf{D}_\varepsilon(v)-\operatorname{div}_x\mathbf{F}_\varepsilon(A)) \text{ is bounded in } L^1_{loc}(\mathbf{R}_+,W^{-s,1}_{loc}(\mathbf{R}^3))$$

(Indeed, we recall that $\mathbf{D}_\varepsilon(v)\to 0$ while $\mathbf{F}_\varepsilon(A)$ is bounded in $L^1_{loc}(\mathbf{R}_+\times\mathbf{R}^3)$.).

Together with the compactness in the x-variable that follows from velocity averaging, this implies that

$$P\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle\to u \text{ in } L^2_{loc}(\mathbf{R}_+\times\mathbf{R}^3)\,.$$

We also recall that

$$\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle\to u \text{ weakly in } L^2_{loc}(\mathbf{R}_+\times\mathbf{R}^3)\,.$$

However, we *do not* seek to prove that

$$\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle\to u \text{ strongly in } L^2_{loc}(\mathbf{R}_+\times\mathbf{R}^3)\,.$$

Instead, we prove that

$$P\operatorname{div}_x\left(\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle^{\otimes 2}\right)\to P\operatorname{div}_x\left(u^{\otimes 2}\right) \text{ in } \mathscr{D}'(\mathbf{R}^*_+\times\mathbf{R}^3) \text{ as } \varepsilon\to 0\,.$$

This is discussed in detail in section 7.2.3 of [40]. Observe that

$$\begin{aligned}\varepsilon\partial_t\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle+\nabla_x\langle\tfrac13|v|^2_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle\to 0 \text{ in } L^1_{loc}(\mathbf{R}_+;W^{-1,1}_{loc}(\mathbf{R}^3))\,,\\ \varepsilon\partial_t\langle\tfrac13|v|^2_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle+\operatorname{div}_x\langle\tfrac53 v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle\to 0 \text{ in } L^1_{loc}(\mathbf{R}_+;W^{-1,1}_{loc}(\mathbf{R}^3))\,,\end{aligned}$$

as $\varepsilon \to 0$. Setting $\nabla_x \pi_\varepsilon = (I-P)\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$, the system above becomes

$$\begin{aligned} \varepsilon \partial_t \nabla_x \pi_\varepsilon + \nabla_x \langle \tfrac{1}{3}|v|^2_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \to 0 \text{ in } L^1_{loc}(\mathbf{R}_+; W^{-s,1}_{loc}(\mathbf{R}^3)) , \quad s > 1 , \\ \varepsilon \partial_t \langle \tfrac{1}{3}|v|^2_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle + \tfrac{5}{3}\Delta_x \pi_\varepsilon \to 0 \text{ in } L^1_{loc}(\mathbf{R}_+; W^{-1,1}_{loc}(\mathbf{R}^3)) . \end{aligned}$$

At this point, we apply the following elegant observation.

Lemma 3.8 (P.-L. Lions-N. Masmoudi [56]) *Let $c \neq 0$ and let ϕ_ε and $\nabla_x \psi_\varepsilon$ be bounded families in $L^\infty_{loc}(\mathbf{R}_+; L^2_{loc}(\mathbf{R}^3))$ such that*

$$\begin{cases} \partial_t \phi_\varepsilon + \dfrac{1}{\varepsilon}\Delta_x \psi_\varepsilon = \dfrac{1}{\varepsilon}\Phi_\varepsilon , \\ \partial_t \nabla_x \psi_\varepsilon + \dfrac{c^2}{\varepsilon}\nabla_x \phi_\varepsilon = \dfrac{1}{\varepsilon}\nabla \Psi_\varepsilon , \end{cases}$$

where

$$\Phi_\varepsilon \text{ and } \nabla \Psi_\varepsilon \to 0 \text{ strongly in } L^1_{loc}(\mathbf{R}_+; L^2_{loc}(\mathbf{R}^3))$$

as $\varepsilon \to 0$. Then

$$P \operatorname{div}_x((\nabla_x \psi_\varepsilon)^{\otimes 2}) \text{ and } \operatorname{div}_x(\phi_\varepsilon \nabla_x \psi_\varepsilon) \to 0$$

in the sense of distributions on $\mathbf{R}^*_+ \times \mathbf{R}^3$ as $\varepsilon \to 0$.

In view of the uniform in time modulus of L^2 continuity (11), the Lions-Masmoudi argument can be applied with π_ε in the place of ψ_ε after regularization in the variable x. Eventually, one finds that

$$P \operatorname{div}_x((\nabla_x \pi_\varepsilon)^{\otimes 2}) \to 0 \text{ in } \mathscr{D}'(\mathbf{R}^*_+ \times \mathbf{R}^3) .$$

On the other hand, the limiting velocity field is divergence-free and therefore

$$\nabla_x \pi_\varepsilon \to 0 \text{ weakly in } L^2_{loc}(\mathbf{R}_+ \times \mathbf{R}^3) \text{ as } \varepsilon \to 0 .$$

Splitting

$$\begin{aligned} P \operatorname{div}_x \left(\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle^{\otimes 2} \right) = P \operatorname{div}_x \left((P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle)^{\otimes 2} \right) + P \operatorname{div}_x \left(P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \otimes \nabla_x \pi_\varepsilon \right) \\ + P \left(\nabla_x \pi_\varepsilon \otimes P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \right) + P \operatorname{div}_x \left((\nabla_x \pi_\varepsilon)^{\otimes 2} \right) \end{aligned}$$

The last two terms vanish with ε while the first converges to $P \operatorname{div}_x(u^{\otimes 2})$ since $P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \to u$ strongly in $L^2_{loc}(dtdx)$.

The interested reader is referred to section 7.3.2 of [40] for the missing details.

3.4 The Key Uniform Integrability Estimates

Eventually, in view of the discussion above, everything is reduced to obtaining the uniform integrability of the family

$$\left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2(1+|v|) \quad \text{on } [0,T]\times K\times\mathbf{R}^3\,,$$

which is the main objective of the present section, stated in the proposition below. This is a (slightly easier) variant of some analogous control on the relative fluctuations of distribution function, identified but left unverified in [9].

Proving this uniform integrability statement remained the main obstruction in deriving Leray solutions of the Navier-Stokes equation from renormalized solutions of the Boltzmann equation, after a sequence of important steps in the understanding of the limit, such as [56] (which explained how to handle oscillations in the time variable), and [10, 32] which reduced the task of controlling conservation defects to the uniform integrability result stated below.

Therefore, obtaining this uniform integrability property remained the only missing step for a complete proof of the incompressible Navier-Stokes limit of the Boltzmann equation. The arguments leading to this uniform integrability property were eventually found in [38]. They involved a refinement of velocity averaging techniques adapted to the L^1 setting ([37]).

Proposition 3.9 (F. Golse-L. Saint-Raymond [38, 40]) *For each $T>0$ and each compact $K\subset\mathbf{R}^3$, the family $\left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2(1+|v|)$ is uniformly integrable on the set $[0,T]\times K\times\mathbf{R}^3$.*

This proposition is really the core of the proof of the incompressible Navier-Stokes limit of the Boltzmann equation in [38, 40]. It involves two main ideas.

3.4.1 Idea no. 1: Uniform Integrability in the v Variable

First we must define this notion of "uniform integrability in one variable" for functions of several variables.

Definition 3.10 *A family of functions $\phi_\varepsilon\equiv\phi_\varepsilon(x,y)\in L^1_{x,y}(d\mu(x)d\nu(y))$ is uniformly integrable in the y-variable for the measure $\mu\otimes\nu$ if and only if*

$$\int\sup_{\nu(A)<\alpha}\int_A|\phi_\varepsilon(x,y)|d\nu(y)d\mu(x)\to 0 \textit{ as } \alpha\to 0 \textit{ uniformly in } \varepsilon\,.$$

The following observation is a first step in the proof of the proposition above.

Lemma 3.11 *The family*

$$\left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2(1+|v|)$$

is uniformly integrable in the v variable on $[0,T]\times K\times\mathbf{R}^3$ *for the measure* $dtdxMdv$.

This is Proposition 3.2 in [40] (see also Lemma 3.1 in that same reference).

Proof (Sketch of the proof). Start from the formula

$$\mathscr{L}\left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)=\varepsilon\mathscr{Q}\left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon},\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)-\frac{1}{\varepsilon}\mathscr{Q}\left(\sqrt{G_\varepsilon},\sqrt{G_\varepsilon}\right),$$

and use the bound [36]

$$\|\mathscr{Q}(f,f)\|_{L^2((1+|v|)^{-1}Mdv)}\le C\|f\|_{L^2(Mdv)}\|f\|_{L^2((1+|v|)Mdv)}.$$

This leads to the following estimate:

$$\begin{aligned}\left(1-O(\varepsilon)\left\|\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right\|_{L^2(Mdv)}\right)\left\|\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}-\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right\|_{L^2((1+|v|)Mdv)}&\\ \le O(\varepsilon)_{L^2_{t,x}}+O(\varepsilon)\left\|\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right\|^2_{L^2(Mdv)}&.\end{aligned}$$

This estimates tells us that the quantity $\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}$ stays close to its associated infinitesimal Maxwellian, which is both smooth and repidly decaying in the variable v.

3.4.2 Idea no. 2: a L^1 Variant of Velocity Averaging

The exact analogue of the velocity averaging theorem (Theorem 3.7) above would be the following statement:

"Let G_n be a bounded sequence in $L^1(\mathbf{R}^N_x\times\mathbf{R}^N_v)$ such that $S_n:=v\cdot\nabla_x G_n$ is bounded in $L^1(\mathbf{R}^N_x\times\mathbf{R}^N_v)$. Then the sequence $\mathscr{A}_\phi[G_n]$ is strongly relatively compact in $L^1_{loc}(\mathbf{R}^N_x)$ for each $\phi\in C_b(\mathbf{R}^N)$."

Unfortunately, this statement is wrong, as shown by the following counterexample (see counterexample 1 in [34]).

Let $N>1$ and let $\psi\in C_c^\infty(\mathbf{R}^N)$ satisfy

$$\psi\ge 0\text{ on }\mathbf{R}^N,\quad\text{ and }\int_{\mathbf{R}^N}\psi(z)dz=1.$$

Let $v_0\neq 0$, and consider the sequence $\Psi_n(x,v)=n^{2N}\psi(nx)\psi(n(v-v_0))$. Obviously

$$\|\Psi_n\|_{L^1(\mathbf{R}^N\times\mathbf{R}^N)}=1,\quad\text{ and }\Psi_n\to\delta_{(0,v_0)}\text{ in }\mathscr{D}'(\mathbf{R}^N\times\mathbf{R}^N)$$

as $n\to\infty$. Let $\Phi_n\equiv\Phi_n(x,v)$ be defined by the formula

$$\Phi_n(x,v) := \int_0^\infty e^{-t}\Psi_n(x-tv,v)dt\,,$$

so that

$$\Phi_n + v\cdot\nabla_x\Phi_n = \Psi_n\,.$$

In particular, one has

$$\|\Phi_n\|_{L^1(\mathbf{R}^N\times\mathbf{R}^N)} \leq 1\,,\text{ so that } \|v\cdot\nabla_x\Phi_n\|_{L^1(\mathbf{R}^N\times\mathbf{R}^N)} \leq 2\,.$$

Yet the explicit formula above for Φ_n shows that $\mathscr{A}_1[\Phi_n] \to \mu$ in $\mathscr{D}'(\mathbf{R}^N\times\mathbf{R}^N)$ as $n\to\infty$, where μ is the Radon measure defined by the formula

$$\langle\mu,\chi\rangle := \int_0^\infty e^{-t}\chi(-tv_0)dt\,.$$

In particular, μ is a Borel probability measure concentrated on a half-line, which is therefore not absolutely continuous with respect to the Lebesgue measure if $N\geq 2$. This excludes the possibility that any subsequence of $\mathscr{A}_1[\Phi_n]$ might converge in $L^1_{loc}(\mathbf{R}^N)$ for the strong topology.

The appropriate generalization to the L^1 setting of the velocity averaging theorem is as follows.

Theorem 3.12 (F. Golse-L. Saint-Raymond [37]) *Let $f_n \equiv f_n(x,v)$ be a bounded sequence in $L^1_{loc}(\mathbf{R}^N\times\mathbf{R}^N)$ such that $v\cdot\nabla_x f_n$ is also bounded in $L^1_{loc}(\mathbf{R}^N\times\mathbf{R}^N)$. Assume that f_n is locally uniformly integrable in v. Then*

- *f_n is locally uniformly integrable (in x,v), and*
- *for each test function $\phi\in L^\infty_{comp}(\mathbf{R}^N_v)$, the sequence of averages*

$$\mathscr{A}_\phi[f_n] : x \mapsto \int f_n(x,v)\phi(v)dv$$

is relatively compact in $L^1_{loc}(\mathbf{R}^N)$.

Proof (Main idea in the proof). Let us prove that the sequence of averages $\mathscr{A}_\phi[f_n]$ is locally uniformly integrable. Without loss of generality, one can assume that both $f_n\geq 0$ and $\phi\geq 0$.

Let A be a measurable subset of $\mathbf{R}^N$ of finite Lebesgue measure. Let $\chi\equiv\chi(t,x,v)$ be the solution of the Cauchy problem

$$\partial_t\chi + v\cdot\nabla_x\chi = 0\,,\qquad \chi(0,x,v) = \mathbf{1}_A(x)\,.$$

Clearly the solution χ of this Cauchy problem is of the form $\chi(t,x,v) = \mathbf{1}_{A_x(t)}(v)$. (Indeed, χ takes the values 0 and 1 only). On the other hand,

$$|A_x(t)| = \int_{\mathbf{R}^N}\chi(t,x,v)dv = \int_{\mathbf{R}^N}\mathbf{1}_A(x-tv)dv = \frac{|A|}{t^N}\,.$$

(This is the basic dispersion estimate for the free transport equation.)

Set

$$\begin{cases} g_n(x,v) := f_n(x,v)\phi(v)\,, \text{ and} \\ h_n(x,v) := v\cdot\nabla_x g_n(x,v) = \phi(v)(v\cdot\nabla_x f_n(x,v))\,. \end{cases}$$

Both g_n and h_n are bounded in $L^1(\mathbf{R}^N\times\mathbf{R}^N)$, while g_n is uniformly integrable in v.

Observe next that

$$\int_A\int g_n dvdx = \int_{\mathbf{R}^N}\int_{A_x(t)} g_n dvdx - \int_0^t\iint_{\mathbf{R}^N\times\mathbf{R}^N} h_n(x,v)\chi(s,x,v)dxdvds\,.$$

(To see this, integrate by parts in the second term on the right hand side.)

The second integral on the right hand side is $O(t)\sup\|h_n\|_{L^1(\mathbf{R}^N\times\mathbf{R}^N)}$ and can be made less than ε by choosing $t>0$ small enough. With $t>0$ chosen in this way, observe that $|A_x(t)|\to 0$ as $|A|\to 0$ by the dispersion estimate above. Hence the first integral on the right hand side vanishes by uniform integrability in v.

A preliminary result in this direction was obtained in [67] — see also Proposition 6 in [34] in the case where the assumption of uniform integrability in the v variable is replaced with the assumption of the type

$$f_n + v\cdot\nabla_x f_n \text{ bounded in } L^1(\mathbf{R}^N)(L^p(\mathbf{R}^N)) \text{ with } p>1\,.$$

Conclusion

There are several other problems in the fluid dynamic limits of the kinetic theory of gases which have not been discussed in these lectures.

Boundary value problems are one such class of problems. The theory of renormalized solutions of the boundary value problem for the Boltzmann equation involves significant additional difficulties not present in the case of the Cauchy problem in the whole Euclidian space or in the torus. These difficulties are due to the nonlocal character (in the v variable) of most of the physically relevant boundary conditions in the kinetic theory of gases. The interaction of the renormalization procedure with the boundary condition was fully understood in a rather remarkable paper by S. Mischler [60]. The fluid dynamic limits of boundary value problems for the Boltzmann equation are reviewed in [70] (see also [59] for a thorough discussion of the Stokes limit of the Boltzmann equation in the presence of boundaries). See also [11, 31] for a discussion of the incompressible Euler limit, also in the presence of boundaries.

We also refer to [74] for a discussion of fluid dynamic limits of the Boltzmann equation in the presence of boundaries in terms of a modified analogue of the Hilbert expansion involving various kinds of boundary layer terms. These boundary layers include in particular Knudsen layers, matching the first terms in Hilbert's expansion with the boundary data — which may fail to be compatible with the dependence in the velocity variable of the various terms in Hilbert's expansion. The mathematical theory of Knudsen layers has been treated in a series of articles [6, 21, 76, 12, 29].

There also remain several outstanding open problems in the context of fluid dynamic limits of the kinetic theory of gases.

First, it would be important to have a a proof of the compressible Euler limit of the Boltzmann equation that would not be limited by the regularity of the solution of the target system as in the work of Caflisch or Nishida described in lecture 1. Of course, this would require having an adequate existence theory of global weak solutions of the compressible Euler system. This is of course a formidable problem in itself, which may not necessarily be directly related to kinetic models. At the time of this writing, global existence of weak solutions of the compressible Euler system has been proved in space dimension 1, for all bounded initial data with small total variation, by using Glimm's scheme [27, 58]. Whether such solutions can be obtained as limits of solutions of the Boltzmann equation is a difficult open problem.

Finally, we should mention that fluid dynamic limits of the Boltzmann equation should also be investigated in the regime of steady solutions. These are important for applications, since steady solutions describe flows in a permanent regime. Unfortunately the theory of steady solutions of the Boltzmann equation is much less well understood as that of the evolution problem — see [44, 45, 2]. Formal results on fluid dynamic limits of steady solutions of the Boltzmann equation are discussed in [74].

4 Appendix 1: On Isotropic Tensor Fields

In this section, we have gathered several results bearing on isotropic tensor fields that are used in lectures 1 and 3.

4.1 On the Structure of Isotropic Tensor Fields

Let $T : \mathbf{R}^N \to (\mathbf{R}^N)^{\otimes m}$ be a tensor field on the N-dimensional Euclidian space $\mathbf{R}^N$, endowed with the canonical inner product (i.e. the one for which the canonical basis is orthonormal). The tensor field T is said to be *isotropic* if

$$T(Qv) = Q \cdot T(v), \quad \text{for each } v \in \mathbf{R}^N \text{ and each } Q \in O_N(\mathbf{R}).$$

Here, the notation $A \cdot \tau$ designates the action of the matrix $A \in M_N(\mathbf{R})$ on the tensor $\tau \in (\mathbf{R}^N)^m$ defined by

$$A \cdot (v_1 \otimes \ldots \otimes v_m) = (Av_1) \otimes \ldots \otimes (Av_m).$$

Lemma 4.1 *Let $T : \mathbf{R}^N \to (\mathbf{R}^N)^{\otimes m}$ be an isotropic tensor field on $\mathbf{R}^N$.*

- *If $m = 0$, then T is a radial real-valued function, i.e. T is of the form*

$$T(\xi) = \tau(|\xi|)$$

where τ is a real-valued function defined on $\mathbf{R}_+$.

- *If $m = 1$, then T is of the form*

$$T(\xi) = \tau(|\xi|)\xi\,, \quad \xi \in \mathbf{R}^N\,,$$

where τ is a real-valued function defined on $\mathbf{R}_+$.

- *If $m = 2$ and $T(\xi)$ is symmetric*[3] *for each $\xi \in \mathbf{R}^N$, then T is of the form*

$$T(\xi) = \lambda_1(|\xi|)I + \lambda_2(|\xi|)\xi\,, \quad \xi \in \mathbf{R}^N\,.$$

Proof. We distinguish the cases corresponding to the different values of m.

Case $m = 0$. In that case $T : \mathbf{R}^N \to \mathbf{R}$ satisfies $T(Q\xi) = T(\xi)$ for all $Q \in O_N(\mathbf{R})$. Let e_1 be the first vector in the canonical basis of $\mathbf{R}^N$. For each $\xi \in \mathbf{R}^N$, there exists $Q \in O_N(\mathbf{R})$ such that $Q\xi = |\xi|e_1$. Thus $T(\xi) = T(|\xi|e_1)$ so that T is a function of $|\xi|$ only, i.e. there exists $\tau : \mathbf{R}_+ \to \mathbf{R}$ such that $T(\xi) = \tau(|\xi|)$.

Case $m = 1$. In that case $T : \mathbf{R}^N \to \mathbf{R}^N$ satisfies

$$T(Q\xi) = QT(\xi) \text{ for each } Q \in O_N(\mathbf{R})\,.$$

For $\xi = 0$, specializing the identity above to $Q = -I$, one has $T(0) = -T(0) = 0$.

For $\xi \neq 0$, let Q run through the group $O_N(\mathbf{R})_\xi$ of orthogonal matrices leaving ξ invariant. This group is isomorphic to the set of orthogonal linear transformations on $(\mathbf{R}\xi)^\perp$. Thus, given $\zeta_1 \neq \zeta_2 \in (\mathbf{R}\xi)^\perp$ such that $|\zeta_1| = |\zeta_2|$, there exists $Q \in O_N(\mathbf{R})_\xi$ such that $Q\zeta_1 = \zeta_2$, i.e. the subgroup $O_N(\mathbf{R})_\xi$ acts transitively on $(\mathbf{R}\xi)^\perp$. Since

$$QT(\xi) = T(\xi) \text{ for each } Q \in O_N(\mathbf{R})_\xi\,,$$

one has

$$Q\left(T(\xi) - (e_\xi \cdot T(\xi))e_\xi\right) = T(\xi) - (e_\xi \cdot T(\xi))e_\xi \text{ for each } Q \in O_N(\mathbf{R})$$

and since $T(\xi) - (e_\xi \cdot T(\xi))e_\xi \perp \xi$ we conclude that

$$T(\xi) - (e_\xi \cdot T(\xi))e_\xi = 0\,.$$

In other words, $T(\xi) = t(\xi)\xi$ for all $\xi \neq 0$, with $t(Q\xi) = \tau(Q\xi)$ for all $\xi \in \mathbf{R}^N$ and $Q \in O_N(\mathbf{R})$. One conclude with the result for the case $m = 0$.

Case $m = 2$ First we use the canonical identification $(\mathbf{R}^N)^{\otimes 2} \simeq M_N(\mathbf{R})$ defined by the formula $(v \otimes w)\xi := (w \cdot \xi)v$ for each $v, w, \xi \in \mathbf{R}^N$. Therefore, $Q \cdot (v \otimes w) = (Qv) \otimes (Qw)$ is identified with $Q(v \otimes w)Q^T$.

---

[3] Consider the endomorphism of $(\mathbf{R}^N)^{\otimes 2}$ defined by

$$u \otimes v \mapsto (u \otimes v)^\sigma = v \otimes u\,.$$

An element T of $(\mathbf{R}^N)^{\otimes 2}$ is said to be symmetric if and only if $T^\sigma = T$.

With this identification $T : \mathbf{R}^N \to (\mathbf{R}^N)^{\otimes 2}$ satisfies

$$T(\xi) = T(\xi)^T \text{ and } T(Q\xi) = Q \cdot T(\xi) = QT(\xi)Q^T \text{ for each } Q \in O_N(\mathbf{R}).$$

The case $\xi = 0$ is obvious: the symmetric matrix with real entries $T(0)$ satisfies $T(0) = QT(0)Q^T$ for all $Q \in O_N(\mathbf{R})$. Since $T(0)$ is diagonalizable and possesses an orthonormal basis of eigenvectors, $T(0)$ must be diagonal (take Q to be the matrix whose columns form an orthonormal basis of eigenvectors of $T(0)$). If $T(0)$ is not of the form λI, let u and v to be unitary eigenvectors of $T(0)$ associated to different eigenvalues, taking Q to be a rotation of an angle $\pm\frac{\pi}{4}$ in the plane leads to a contradiction, since $QT(0)Q^T$ is not diagonal.

Let $\xi \neq 0$, and consider the vector field S defined by $S(\xi) := T(\xi) \cdot \xi$ for each $\xi \in \mathbf{R}^N$. Since

$$S(R\xi) = RT(\xi)R^T R\xi = RT(\xi)\xi = RS(\xi),$$

the result already established in the case $m = 1$ implies that S is of the form

$$S(\xi) = \alpha(|\xi|)\xi, \quad \xi \in \mathbf{R}^N.$$

Since $T(\xi)$ is identified with a symmetric matrix with real entries and ξ is an eigenvector of $T(\xi)$, the space $(\mathbf{R}\xi)^\perp$ is stable under $T(\xi)$, and can be decomposed as an orthogonal direct sum of eigenspaces of $T(\xi)$. On the other hand, since

$$QT(\xi) = T(\xi)Q \text{ for each } Q \in O_N(\mathbf{R})_\xi,$$

each eigenspace of $T(\xi)$ is stable under Q for each $Q \in O_N(\mathbf{R})_\xi$. Since $O_N(\mathbf{R})_\xi$ acts transitively on $(\mathbf{R}\xi)^\perp$, this implies that $(\mathbf{R}^\xi)^\perp$ is itself an eigenspace of $T(\xi)$. Therefore, appealing to the result already proved in the case $m = 0$, one finds that T is of the form

$$T(\xi) = \alpha(|\xi|)e_\xi \otimes e_\xi + \beta(|\xi|)(I - e_\xi \otimes e_\xi).$$

4.2 Isotropic Tensors and Rotation Invariant Averages of Monomials

We first recall an almost trivial result.

Lemma 4.2 *Let $\chi \equiv \chi(|v|)$ be a measurable radial function defined a.e. on $\mathbf{R}^N$ and such that*

$$\int_{\mathbf{R}^N} |\chi(|v|)||v|^2 dv < \infty.$$

Then, for all $i, j = 1, \ldots, N$, one has

$$\int_{\mathbf{R}^N} |\chi(|v|) v_i v_j dv = \frac{1}{N}\delta_{ij} \int_{\mathbf{R}^N} |\chi(|v|)||v|^2 dv.$$

Proof. Let $\chi \equiv \chi(|v|)$ be a measurable radial function defined a.e. on $\mathbf{R}^N$ and such that

$$\int_{\mathbf{R}^N} |\chi(|v|)||v|^2 dv < \infty .$$

Set

$$T_{i,j} := \int_{\mathbf{R}^N} \chi(|v|) v_i v_j dv , \quad i,j = 1,\ldots,N .$$

Consider the vector field T defined on $\mathbf{R}^N$ by the formula

$$T(\xi) := \int_{\mathbf{R}^N} \chi(|v|)(v\cdot\xi) v dv ,$$

or equivalently

$$T(\xi)_i := \sum_{j=1}^{N} T_{ij}\xi_j .$$

Obviously, for each $R \in O_N(\mathbf{R})$, one has

$$\begin{aligned} T(R\xi) = \int_{\mathbf{R}^N} \chi(|v|)(v\cdot R\xi) v dv &= \int_{\mathbf{R}^N} \chi(|v|)(R^T v\cdot\xi) v dv \\ &= \int_{\mathbf{R}^N} \chi(|w|)(w\cdot\xi) R w dw = RT(\xi) , \end{aligned}$$

where the third equality follows from the substitution $w = R^T v$ in the integral. By Lemma 4.1, T is of the form

$$T(\xi) = \tau(|\xi|)\xi ,$$

and since T is obviously linear in ξ, the function τ is a constant, so that

$$T(\xi) = \tau\xi ,$$

or equivalently

$$T_{ij} = \tau\delta_{ij} .$$

In particular

$$N\tau = \sum_{i=1}^{N} T_{ii} = \int_{\mathbf{R}^N} \chi(|v|)|v|^2 dv ,$$

which gives the formula for τ.

Of course, one could also have observed that the matrix with entries

$$\int_{\mathbf{R}^N} \chi(|v|) v_i v_j dv$$

for $i,j = 1,\ldots,N$ is real and symmetric, and commutes with every orthogonal matrix. As already explained in the proof of Lemma 4.1 (case $m = 2$ and $\xi = 0$), such a matrix is proportional to the identity matrix.

However, the (slightly) more complicated proof given above is easily generalized to the case of rotation invariant averages of quartic monomials discussed below.

Lemma 4.3 *Let $\chi \equiv \chi(|v|)$ be a measurable radial function defined a.e. on $\mathbf{R}^N$ and such that*

$$\int_{\mathbf{R}^N} |\chi(|v|)||v|^4 dv < \infty .$$

Set

$$T_{ijkl} := \int_{\mathbf{R}^N} \chi(|v|) v_i v_j v_k v_l dv , \quad i,j,k,l = 1,\dots,N .$$

Then T_{ijkl} is of the form

$$T_{ijkl} := t_0(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) ,$$

where

$$t_0 = \tfrac{1}{N(N+2)} \int_{\mathbf{R}^N} \chi(|v|)|v|^4 dv .$$

Proof. Consider the map T defined by

$$T : \mathbf{R}^N \ni \xi \mapsto \int_{\mathbf{R}^N} \chi(|v|)(\xi \cdot v)^2 v \otimes v dv \in (\mathbf{R}^N)^{\otimes 2} .$$

Obviously $T(\xi)$ is a symmetric tensor (as an integral linear combination of symmetric tensors $v \otimes v$) and

$$T(\xi) = \sum_{k,l} T_{ijkl} \xi_k \xi_l e_i \otimes e_j$$

where e_i is the ith vector of the canonical basis of $\mathbf{R}^N$, or equivalently

$$T(\xi)_{ij} = \sum_{k,l} T_{ijkl} \xi_k \xi_l .$$

Moreover, for each $R \in O_N(\mathbf{R})$, one has

$$\begin{aligned} T(R\xi) &= \int_{\mathbf{R}^N} \chi(|v|)(R\xi \cdot v)^2 v \otimes v dv \\ &= \int_{\mathbf{R}^N} \chi(|v|)(\xi \cdot R^T v)^2 v \otimes v dv \\ &= \int_{\mathbf{R}^N} \chi(|w|)(\xi \cdot w)^2 (Rw) \otimes (Rw) dw = RT(\xi)R^T = R \cdot T(\xi) , \end{aligned}$$

where the third equality follows from the substitution $w = R^T v$ in the integral.

In other words, T is an isotropic symmetric tensor field of order 2, and is therefore of the form

$$T(\xi) = \tau_0(|\xi|)I + \tau_1(|\xi|)\xi \otimes \xi .$$

Besides, T is quadratic in ξ, which implies that $\tau_0(|\xi|) = t_0|\xi|^2$ while $\tau_1(|\xi|) = t_1$ is a constant. Finally

$$T(\xi) = t_0|\xi|^2 I + t_1 \xi \otimes \xi \,.$$

In particular, T is of class C^∞ on $\mathbf{R}^N$, and one has

$$2T_{ijpq} = \frac{\partial^2}{\partial\xi_p\partial\xi_q} T(\xi)_{ij} = 2t_0\delta_{pq}\delta_{ij} + t_1(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) \,.$$

Since $T_{ijpq} = T_{ipjq}$, one has $t_1 = 2t_0$.

Finally

$$\int_{\mathbf{R}^N} \chi(|v|)|v|^4 dv = \sum_{i,k=1}^N T_{ikik} = t_0 \sum_{i,k=1}^N (\delta_{ik}\delta_{ik} + \delta_{ii}\delta_{kk} + \delta_{ik}\delta_{ik}) = t_0 N(N+2) \,,$$

which concludes the proof.

5 Appendix 2: Invariance Properties of the Linearized Collision Integral and Applications

For all $\rho,\theta > 0$ and $u \in \mathbf{R}^3$, we designate by $\mathscr{L}_{\rho,u,\theta}$ the linearization at $\mathscr{M}_{(\rho,u,\theta)}$ of the Boltzmann collision integral, i.e.

$$\begin{aligned} &\mathscr{L}_{\rho,u,\theta}\phi(v) \\ &:= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(v) + \phi(v_*) - \phi(v') - \phi(v'_*))|(v - v_*)\cdot\omega|\mathscr{M}_{(\rho,u,\theta)}(v_*) dv_* d\omega \,. \end{aligned}$$

First we examine the translation and scale invariance of the linearized collision operator.

Lemma 5.1 *For all $u \in \mathbf{R}^3$ and $\lambda > 0$ denote τ_u and μ_λ the translation and scaling transformations defined by*

$$\tau_u z := z + u \,, \quad \text{and } \mu_\lambda z := \lambda z \,.$$

Then, for each $\phi \in \mathrm{Dom}(\mathscr{L}_{\rho,u,\theta})$, the function $\phi\circ\tau_u\circ\mu_{\sqrt{\theta}}$ belongs to $\mathrm{Dom}(\mathscr{L}_{1,0,1})$ and one has

$$(\mathscr{L}_{\rho,u,\theta}\phi)\circ\tau_u\circ\mu_{\sqrt{\theta}} = \rho\sqrt{\theta}\mathscr{L}_{1,0,1}(\phi\circ\tau_u\circ\mu_{\sqrt{\theta}}) \,.$$

Proof. Since $\mathscr{M}_{(\rho,u,\theta)} = \rho\mathscr{M}_{(1,u,\theta)}$, one has

$$\mathscr{L}_{\rho,u,\theta} = \rho\mathscr{L}_{1,u,\theta} \,.$$

Next, observe (by direct inspection on the formulas (1)) that

$$\begin{cases} v'(v+u, v_*+u, \omega) = v'(v, v_*, \omega) + u\,, \\ v'_*(v+u, v_*+u, \omega) = v'_*(v, v_*, \omega) + u\,. \end{cases}$$

Since the Lebesgue measure is invariant by translation

$$\begin{aligned} &(\mathscr{L}_{1,u,\theta}\phi)(v+u) \\ &= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(v+u) + \phi(w_*) - \phi(v'(v+u, w_*, \omega)) - \phi(v'_*(v+u, w_*, \omega))) \\ &\qquad |(v+u-w_*)\cdot\omega| \mathscr{M}_{(1,u,\theta)}(w_*) dw_* d\omega \\ &= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(v+u) + \phi(w_*) - \phi(v'(v+u, w_*, \omega)) - \phi(v'_*(v+u, w_*, \omega))) \\ &\qquad |(v+u-w_*)\cdot\omega| \mathscr{M}_{(1,0,\theta)}(w_*-u) dw_* d\omega \\ &= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(v+u) + \phi(v_*+u) - \phi(v'(v+u, v_*+u, \omega)) - \phi(v'_*(v+u, v_*+u, \omega))) \\ &\qquad |(v-v_*)\cdot\omega| \mathscr{M}_{(1,0,\theta)}(v_*) dv_* d\omega \\ &= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(v+u) + \phi(v_*+u) - \phi(v'(v, v_*, \omega)+u) - \phi(v'_*(v, v_*, \omega)+u)) \\ &\qquad |(v-v_*)\cdot\omega| \mathscr{M}_{(1,0,\theta)}(v_*) dv_* d\omega \\ &= \mathscr{L}_{1,0,\theta}(\phi\circ\tau_u)(v) \end{aligned}$$

so that

$$(\mathscr{L}_{1,u,\theta}\phi)\circ\tau_u = \mathscr{L}_{1,0,\theta}(\phi\circ\tau_u)\,.$$

Finally, observing that the map $(v, v_*) \mapsto (v'(v, v_*, \omega), v'_*(v, v_*, \omega))$ is homogeneous of degree 1 for each $\omega \in \mathbf{S}^2$ (see formulas (1)), one has

$$\begin{aligned} &\mathscr{L}_{1,0,\theta}\phi(\sqrt{\theta}v) \\ &= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(\sqrt{\theta}v) + \phi(w_*) - \phi(v'(\sqrt{\theta}v, w_*, \omega)) - \phi(v'_*(\sqrt{\theta}v, w_*, \omega))) \\ &\qquad |(\sqrt{\theta}v - w_*)\cdot\omega| \mathscr{M}_{(1,0,\theta)}(w_*) dw_* d\omega \\ &= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(\sqrt{\theta}v) + \phi(w_*) - \phi(v'(\sqrt{\theta}v, w_*, \omega)) - \phi(v'_*(\sqrt{\theta}v, w_*, \omega))) \\ &\qquad |(\sqrt{\theta}v - w_*)\cdot\omega| \mathscr{M}_{(1,0,1)}(w_*/\sqrt{\theta})\theta^{-3/2} dw_* d\omega \\ &= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(\sqrt{\theta}v) + \phi(\sqrt{\theta}v_*) - \phi(v'(\sqrt{\theta}v, \sqrt{\theta}v_*, \omega)) - \phi(v'_*(\sqrt{\theta}v, \sqrt{\theta}v_*, \omega))) \\ &\qquad |(\sqrt{\theta}v - \sqrt{\theta}v_*)\cdot\omega| \mathscr{M}_{(1,0,1)}(v_*) dv_* d\omega \\ &= \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(\sqrt{\theta}v) + \phi(\sqrt{\theta}v_*) - \phi(\sqrt{\theta}v'(v, v_*, \omega)) - \phi(\sqrt{\theta}v'_*(v, v_*, \omega))) \\ &\qquad \sqrt{\theta}|(v-v_*)\cdot\omega| \mathscr{M}_{(1,0,1)}(v_*) dv_* d\omega \\ &= \mathscr{L}_{1,0,1}(\phi\circ\mu_{\sqrt{\theta}})(v) \end{aligned}$$

so that

$$(\mathscr{L}_{1,0,\theta}\phi)\circ\mu_{\sqrt{\theta}} = \sqrt{\theta}\mathscr{L}_{1,0,1}(\phi\circ\mu_{\sqrt{\theta}})\,.$$

The previous lemma shows that we can restrict our attention to $\mathscr{L}_{1,0,1}$, henceforth denoted by $\mathscr{L}$ for simplicity, as in the main body of this text. Then we discuss the invariance of $\mathscr{L}$ under orthogonal transformations.

Lemma 5.2 *For each $R \in O_3(\mathbf{R})$ and each $\phi \in \mathrm{Dom}(\mathscr{L})$, the function $\phi \circ R$ also belongs to* $\mathrm{Dom}(\mathscr{L})$ *and one has*

$$(\mathscr{L}\phi)\circ R = \mathscr{L}(\phi\circ R)\,.$$

Proof. Let $R \in O_3(\mathbf{R})$ and $\phi \equiv \phi(v)$ be an element of $\mathrm{Dom}(\mathscr{L})$. Then, elementary changes of variables show that

$$\begin{aligned}
&\mathscr{L}\phi(Rv)\\
&= \iint_{\mathbf{R}^3\times\mathbf{S}^2}(\phi(Rv)+\phi(w_*)-\phi(v'(Rv,w_*,u))-\phi(v'_*(Rv,w_*,u)))\\
&\qquad |(Rv-w_*)\cdot u|M(w_*)dw_*du\\
&= \iint_{\mathbf{R}^3\times\mathbf{S}^2}(\phi(Rv)+\phi(Rv_*)-\phi(v'(Rv,Rv_*,u))-\phi(v'_*(Rv,Rv_*,u)))\\
&\qquad |(Rv-Rv_*)\cdot u|M(Rv_*)dv_*du\\
&= \iint_{\mathbf{R}^3\times\mathbf{S}^2}(\phi(Rv)+\phi(Rv_*)-\phi(v'(Rv,Rv_*,R\omega))-\phi(v'_*(Rv,Rv_*,R\omega)))\\
&\qquad |(Rv-Rv_*)\cdot R\omega|M(v_*)dv_*d\omega\\
&= \iint_{\mathbf{R}^3\times\mathbf{S}^2}(\phi(Rv)+\phi(Rv_*)-\phi(v'(Rv,Rv_*,R\omega))-\phi(v'_*(Rv,Rv_*,R\omega)))\\
&\qquad |(v-v_*)\cdot \omega|M(v_*)dv_*d\omega\,.
\end{aligned}$$

Formulas (1) show that

$$\left\{\begin{aligned} v'(Rv,Rv_*,R\omega) &= Rv'(v,v_*,\omega)\,,\\ v'_*(Rv,Rv_*,R\omega) &= Rv'_*(v,v_*,\omega)\,.\end{aligned}\right.$$

Therefore, the computation above implies that

$$(\mathscr{L}\phi)\circ R = \mathscr{L}(\phi\circ R)\,.$$

Next we define the functions α and β used in the computation of the viscosity and heat diffusion in the compressible Navier-Stokes system — see lecture 1, especially formulas (6).

Lemma 5.3 *For each $i,j,k = 1,2,3$, one has A_{ij} and $B_k \in \mathrm{Ran}\,\mathscr{L}$.*

Proof. First observe that

$$A_{ij} \perp \operatorname{Ker} \mathscr{L}\,, \quad \text{and } B_k \perp \operatorname{Ker} \mathscr{L}\,.$$

for each $i, j, k = 1, 2, 3$.

The orthogonality relations

$$A_{ij} \perp v_k\,, \quad B_k \perp 1\,, \quad \text{and } B_k \perp |v|^2\,, \quad \text{for all } i, j, k = 1, 2, 3$$

are obvious, since the corresponding inner products are integrals of odd summable functions on $\mathbf{R}^3$. That

$$A_{ij} \perp 1 \text{ and } A_{ij} \perp |v|^2\,, \quad \text{for all } i, j, k = 1, 2, 3$$

follows from Lemma 4.2. Indeed, for each measurable radial function $\phi \equiv \phi(|v|)$ such that

$$\int_{\mathbf{R}^3} |\phi(|v|)||v|^4 M(v)dv < \infty\,,$$

one has

$$\int_{\mathbf{R}^3} \phi(|v|) A_{ij}(v) M(v) dv = c\delta_{ij}$$

by Lemma 4.2, and

$$c = \tfrac{1}{3} \int_{\mathbf{R}^3} \phi(|v|) \operatorname{trace}(A(v)) M(v) dv = 0\,.$$

Finally

$$\int_{\mathbf{R}^3} v_i B_j(v) M(v) dv = \int_{\mathbf{R}^3} v_i v_j (|v|^2 - 5) M(v) dv = c' \delta_{ij}$$

again by Lemma 4.2 and a straightfoarward computation shows that

$$c' = \tfrac{1}{3} \int_{\mathbf{R}^3} (|v|^4 - 5|v|^2) M(v) dv = 0\,.$$

Since $\mathscr{L}$ is a self-adjoint Fredholm operator on $L^2(\mathbf{R}^3, Mdv)$ with null space

$$\operatorname{Ker} \mathscr{L} = \operatorname{span}(\{1, v_1, v_2, v_3, |v|^2\})$$

by Hilbert's theorem (Theorem 3.2), the orthogonality properties above imply that

$$A_{ij} \text{ and } B_k \in \operatorname{Ran} \mathscr{L}\,.$$

Lemma 5.4 *Let $\hat{A}$ be the unique symmetric tensor field of order 2 on $\mathbf{R}^3$ such that $\hat{A}_{ij} \in \operatorname{Dom} \mathscr{L} \cap (\operatorname{Ker} \mathscr{L})^\perp$ for all $1 \le i, j \le 3$ and*

$$\mathscr{L} \hat{A}_{ij} = A_{ij}\,, \quad 1 \le i, j \le 3\,.$$

Then, there exists a radial measurable function $\alpha \equiv \alpha(|v|)$ defined on $\mathbf{R}^3$ such that

$$\hat{A}(v) = \alpha(|v|)A(v)\,, \quad \textit{for a.e. } v \in \mathbf{R}^3\,.$$

Likewise, let $\hat{B}$ *be the unique vector field on* $\mathbf{R}^3$ *such that, for each* $i = 1,2,3$*, one has* $\hat{B}_i \in \mathrm{Dom}\,\mathscr{L} \cap (\mathrm{Ker}\,\mathscr{L})^{\perp}$ *and*

$$\mathscr{L}\hat{B}_i = B_i\,, \quad 1 \le i \le 3\,.$$

Then, there exists a radial measurable function $\beta \equiv \beta(|v|)$ *defined on* $\mathbf{R}^3$ *such that*

$$\hat{B}(v) = \beta(|v|)B(v)\,, \quad \textit{for a.e. } v \in \mathbf{R}^3\,.$$

Proof. Applying this identity to each component of $\hat{A} \in \mathrm{Dom}\,\mathscr{L} \cap (\mathrm{Ker}\,\mathscr{L})^{\perp}$ such that

$$\mathscr{L}\hat{A} = A \text{ componentwise}$$

shows that

$$\mathscr{L}(\hat{A} \circ R) = (\mathscr{L}\hat{A}) \circ R = A \circ R = R \cdot A = RAR^T = R(\mathscr{L}\hat{A})R^T = \mathscr{L}(R\hat{A}R^T)\,.$$

Since $\hat{A} \circ R$ and $R\hat{A}R^T$ are both orthogonal to $\mathrm{Ker}\,\mathscr{L}$ componentwise, we deduce from Fredholm's alternative that

$$\hat{A} \circ R = R\hat{A}R^T \quad \text{for all } R \in O_3(\mathbf{R})\,.$$

Likewise

$$\hat{A} = \hat{A}^T\,;$$

indeed $\hat{A}$ and $\hat{A}^T \perp \mathrm{Ker}\,\mathscr{L}$ componentwise and $\mathscr{L}(\hat{A} - \hat{A}^T) = A - A^T = 0$, so that $\hat{A} - \hat{A}^T \in \mathrm{Ker}\,\mathscr{L} \cap (\mathrm{Ker}\,\mathscr{L})^T$.

By Lemma 4.1, the tensor field $\hat{A}$ is therefore of the form

$$\hat{A}(v) = \tau_0(|v|)I + \tau_1(|v|)v \otimes v\,.$$

Besides

$$\mathscr{L}(\mathrm{trace}\hat{A}) = \mathrm{trace}(\mathscr{L}\hat{A}) = \mathrm{trace}A = 0 \text{ and } \mathrm{trace}\hat{A} \perp \mathrm{Ker}\,\mathscr{L}\,.$$

Therefore

$$\mathrm{trace}\hat{A} = 3\tau_0(|v|) + |v|^2\tau_1(|v|) = 0\,,$$

which leads to the announced formula for $\hat{A}$.

The case of the integral equation involving the vector field B is treated in the same way. One finds that $\hat{B} \circ R = RB$ for each $R \in O_3(\mathbf{R})$, so that $\hat{B}$ is of the form $\hat{B}(v) = \tau(|v|)v$; the radial function β is defined for all $r \neq \sqrt{5}$ by the formula

$$\beta(r) = \tau(r)/(r^2 - 5)\,.$$

Finally we prove formulas and (7) and Lemma 3.4.

Lemma 5.5 *Let $u \in \mathbf{R}^3$ and $\theta > 0$, and define*

$$A_{u,\theta}(v) := A\left(\frac{v-u}{\sqrt{\theta}}\right), \quad B_{u,\theta}(v) := B\left(\frac{v-u}{\sqrt{\theta}}\right).$$

There exists a unique tensor field $\hat{A}_{u,\theta}$ and a unique vector field $\hat{B}_{u,\theta}$, both belonging to $\operatorname{Dom}\mathscr{L}_{1,u,\theta} \cap (\operatorname{Ker}\mathscr{L}_{1,u,\theta})^{\perp}$ componentwise and such that

$$\mathscr{L}_{1,u,\theta}\hat{A}_{u,\theta} = A_{u,\theta}, \quad \mathscr{L}_{1,u,\theta}\hat{B}_{u,\theta} = B_{u,\theta}.$$

Moreover

$$\begin{cases} \hat{A}_{u,\theta}(v) = \dfrac{1}{\sqrt{\theta}}\alpha\left(\dfrac{|v-u|}{\sqrt{\theta}}\right)A\left(\dfrac{v-u}{\sqrt{\theta}}\right), \\ \hat{B}_{u,\theta}(v) = \dfrac{1}{\sqrt{\theta}}\beta\left(\dfrac{|v-u|}{\sqrt{\theta}}\right)B\left(\dfrac{v-u}{\sqrt{\theta}}\right). \end{cases}$$

Proof. Define

$$\hat{A}_{u,\theta}(v) = \frac{1}{\sqrt{\theta}}\hat{A}\left(\frac{v-u}{\sqrt{\theta}}\right),$$

so that

$$\hat{A}_{u,\theta} \circ \tau_u \circ \mu_{\sqrt{\theta}} = \frac{1}{\sqrt{\theta}}\hat{A}.$$

Using Lemmas 5.4 and 5.1 shows that, if

$$A = \mathscr{L}_{1,0,\theta}(\hat{A}) = \sqrt{\theta}\mathscr{L}_{1,0,\theta}(\hat{A}_{u,\theta} \circ \tau_u \circ \mu_{\sqrt{\theta}}) = (\mathscr{L}_{1,u,\theta}\hat{A}_{u,\theta}) \circ \tau_u \circ \mu_{\sqrt{\theta}}.$$

Equivalently

$$\mathscr{L}_{1,u,\theta}\hat{A}_{u,\theta} = A_{u,\theta},$$

since

$$A_{u,\theta} \circ \tau_u \circ \mu_{\sqrt{\theta}} = A.$$

That $\hat{A}_{u,\theta}\operatorname{Dom}\mathscr{L}_{1,u,\theta} \cap (\operatorname{Ker}\mathscr{L}_{1,u,\theta})^{\perp}$ componentwise is obvious since the tensro field $\hat{A}$ satisfies $\hat{A} \in \operatorname{Dom}\mathscr{L} \cap (\operatorname{Ker}\mathscr{L})^{\perp}$ componentwise.

The case of the vector field $B_{u,\theta}$ is treated in the same manner.

In other words,

$$\begin{cases} \hat{A}_{u,\theta}(v) = \tilde{\alpha}\left(\theta, \dfrac{|v-u|}{\sqrt{\theta}}\right)A\left(\dfrac{v-u}{\sqrt{\theta}}\right) \\ \hat{B}_{u,\theta}(v) = \tilde{\beta}\left(\theta, \dfrac{|v-u|}{\sqrt{\theta}}\right)B\left(\dfrac{v-u}{\sqrt{\theta}}\right), \end{cases}$$

with

$$\tilde{\alpha}(\theta, r) = \frac{1}{\sqrt{\theta}}\alpha(r), \quad \text{and } \tilde{\beta}(\theta, r) = \frac{1}{\sqrt{\theta}}\beta(r).$$

These last formulas and formulas (6) obviously imply formulas (7).

Proof of Lemma 3.4. By Lemma 4.1, for each radial measurable function $\chi \equiv \chi(|v|)$ such that

$$\int_{\mathbf{R}^3} |\chi(|v|)||v|^4 dv < \infty ,$$

By Lemma 4.1

$$\int_{\mathbf{R}^3} \chi(|v|)A_{ij}(v)A_{kl}(v)dv = t_0(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - (2t_1 - t_2)\delta_{ij}\delta_{kl} .$$

In particular

$$\begin{aligned}\sum_{i=1}^{3} \int_{\mathbf{R}^3} \chi(|v|)A_{ij}(v)A_{kl}(v)dv &= \int_{\mathbf{R}^3} \chi(|v|)\,\mathrm{trace}(A(v))A_{kl}(v)dv \\ &= t_0(3\delta_{kl} + \delta_{kl} + \delta_{kl}) - 3(2t_1 - t_2)\delta_{kl}\end{aligned}$$

so that

$$(2t_1 - t_2) = \tfrac{5}{3}t_0 ,$$

and therefore

$$\int_{\mathbf{R}^3} \chi(|v|)A_{ij}(v)A_{kl}(v)dv = t_0(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - \tfrac{2}{3}\delta_{ij}\delta_{kl}) .$$

Thus

$$\begin{aligned}\sum_{i,k=1}^{3} \int_{\mathbf{R}^3} \chi(|v|)A_{ij}(v)^2 dv &= \sum_{i,k=1}^{3} t_0(\delta_{ii}\delta_{kk} + \delta_{ik}\delta_{ik} - \tfrac{2}{3}\delta_{ik}\delta_{ik}) \\ &= t_0(3\cdot 3 + 3 - \tfrac{2}{3}\cdot 3) = 10t_0 .\end{aligned}$$

In particular, with $\chi(|v|) = M(|v|)$, one has

$$\begin{aligned}\langle A_{ij}A_{kl}\rangle &= \tfrac{1}{15}\langle |v|^4\rangle(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - \tfrac{2}{3}\delta_{ij}\delta_{kl}) \\ &= (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - \tfrac{2}{3}\delta_{ij}\delta_{kl}) .\end{aligned}$$

Since $\hat{A}(v) = \alpha(|v|)A(v)$ by Lemma 5.4, one has also

$$\langle \hat{A}_{ij}A_{kl}\rangle = \tfrac{1}{15}\langle \alpha(|v|)|v|^4\rangle(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - \tfrac{2}{3}\delta_{ij}\delta_{kl}) ,$$

which is the sought formula with

$$\nu := \tfrac{1}{15}\langle \alpha(|v|)|v|^4\rangle .$$

Finally

$$\langle \hat{A} : A\rangle = 10\nu = \langle \hat{A} : \mathscr{L}\hat{A}\rangle > 0$$

since $\mathscr{L}$ is a nonnegative operator and $\hat{A} \perp \mathrm{Ker}\,\mathscr{L}$ componentwise. This completes the proof.

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