Towards a new viewpoint on causality for time series

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TOWARDS A NEW VIEWPOINT ON CAUSALITY FOR TIME SERIES

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En hommage amical au Professeur Abdelhaq EL JAI

Abstract. Causation between time series is a most important topic in econometrics, financial engineering, biological and psychological sciences, and many other fields. A new setting is introduced for examining this rather abstract concept. The corresponding calculations, which are much easier than those required by the celebrated Granger-causality, do not necessitate any deterministic or probabilistic modeling. Some convincing computer simulations are presented.

Résumé. La causalité entre chroniques est un sujet capital en économétrie, ingénierie financière, sciences biologiques et psychologiques, et quantité d'autres domaines. On introduit ici une nouvelle approche pour traiter ce concept abstrait. Les calculs, qui sont beaucoup plus simples que ceux liés à la causalité de Granger, bien connue, ne nécessitent aucune modélisation, déterministe ou probabiliste. On présente plusieurs simulations numériques réussies.

1. Introduction

1.1. Generalities

Causality, or the *theory of causation*, is since ever a philosophical mainstay. This is shown by a huge body of writings due to important thinkers like Aristotle, Hume, Kant, Maine de Biran, Mach, Schlick, Meyerson, Carnap, and many others. Let us nevertheless illustrate the difficulty of this concept via the following humorous citation from Bertrand Russell [39]:
The law of causality, I believe, like much that passes muster among philosophers, is a relic of a bygone age, surviving, like the monarchy, only because it is erroneously supposed to do no harm.

1.2. Granger-causality

More recently causality has been investigated via probabilistic tools (see, *e.g.*, Reichenbach [37], Suppes [43], Pearl [36], ...). This is also the case of the Granger-causality [23–25] on time series, to which the names of

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Wiener [44] and Sims [40] are often associated. It has gained a huge popularity which is largely due to the following facts:

- Granger-causality is easy and pragmatic, *i.e.*, it seems to bypass to a large extent any philosophical debate.
- Time is naturally incorporated.

Those attractive features lead to the attribution of the Nobel Prize in economic sciences to Granger in 2003 [26].[1] A time series Y is said there to be a "cause" of a time series X if, and only if, the forecast of X benefits from the knowledge of the past of Y. Granger-causality, which was introduced for answering questions stemming from econometrics and financial engineering and was further developed in many other domains, like, for instance, biology and psychology, has unfortunately not been as successful as many researchers and practitioners hoped. The explanation lies perhaps in its most severe mathematical assumptions, on the

- linear structure of the time series,
- covariance stationarity of the corresponding signals.

They can be only partly weakened via complex operations like *cointegration* (see, *e.g.*, [6, 26, 27]). In spite of several attempts to obtain a nonlinear extension, no theory has been adopted and applied on a large scale to the best of our knowledge.

Remark 1. See [35] for a well written historical account of the classic approach to time series in econometrics, where Granger's works play a key rôle.

1.3. Our approach

This paper suggests a quite different route for examining causality between time series. A theorem by Cartier and Perrin [3] is fundamental. It was already presented in [12] where the connection with financial engineering was developed (see also [13, 15–17]). Let us summarize some important features of this new standpoint:

(1) The existence of *trends*.[2]
(2) The existence of *quick fluctuations*, which yield another setting for classic quantities like volatility [16].
(3) There is no need of a mathematical modeling of the time series. According to our opinion this need might be the key explanation of the difficulties encountered today by the theory of time series.[3]

Uncertainty is then taken into account without the need any probabilistic law. We utilize the definition of *beta* (β) in [15], where some shortcomings of the classic *market*, or *systematic*, *risk* were examined. Introduce for two given time series X, Y their *averaged means* $\mathrm{AV}_\Delta X$, $\mathrm{AV}_\Delta Y$ during a time interval Δ. The quotient

$$\beta_Y^X(\Delta, t) = \frac{\mathrm{AV}_\Delta X}{\mathrm{AV}_\Delta Y}$$

or more precisely, its variation, defines the *relation*, or *influence*,[4] between X and Y at time t. In plain words, the series X and Y are said to be, or not to be, related if the corresponding values of β may be related as follows :

- If $|\beta|$ is *appreciable*,[5] *i.e.*, is neither too small nor too big, and if β has a constant sign during a quite "long" time T, we say that one series is *positively* (resp. *negatively*) *related* to the other during the time lapse T if $\beta > 0$ (resp. $\beta < 0$).
- If the sign of β is changing too often, we say that there is no relation between the series.

[1] Sims also got in 2011 the Nobel prize. Its Nobel lecture on modeling [41] will be discussed elsewhere. See, nevertheless, [12, 17], [14], and Section 1.3.

[2] This is a key assumption in *technical analysis* or *charting* (see, *e.g.*, [1, 32] and the references therein). The notion of trends in the usual time series literature (see, *e.g.*, [22, 30]) does no coincide with ours.

[3] See [14] for a *model-free* control setting, which is most successful from an applied viewpoint.

[4] We are not employing on purpose the polysemic word *cause*, which was at the origin of so many heated controversies!

[5] This terminology is borrowed from nonstandard analysis (see, *e.g.*, [4, 5]).

It might be interesting to

- introduce a different time intervals on the mean averages in order to take into account delays,
- give a more canonical value to β by computing it via *returns*.

The forecast of β thanks to techniques which started in [12] yields moreover a prediction of the relation between the series.

Remark 2. As already stated in [12, 13, 15–17], our approach is connected to recent advances in control engineering and in signal processing.[6] Let us point out therefore that previous works in control have already been employed to analyze some aspects of the theory of causation:

(1) When the differential equations governing a system are known, the control variables, *i.e.*, the causes may be deduced [7].
(2) *Determinism* in discrete-time may be confirmed in the same way as for deterministic ordinary differential equations in continuous time [8, 9].

1.4. **Organization of the paper**

Our paper is organized as follows. Section 2 summarizes our viewpoint on time series, which has already been expounded elsewhere (see, *e.g.*, [12, 17]). Section 3 extracts from [15] the necessary material on the new coefficient β. The academic time series for the numerical experiments displayed in Section 4, which are borrowed from [29], are, as in [28, 29],[7] given by closed-form continuous-time expressions. Some short concluding remarks may be found in Section 5.

2. Time series

2.1. **Nonstandard analysis and the Cartier-Perrin theorem**[8]

Take the time interval $[0, 1] \subset \mathbb{R}$ and introduce as often in *nonstandard analysis* the infinitesimal sampling

$$\mathfrak{T} = \{0 = t_0 < t_1 < \cdots < t_N = 1\}$$

where $t_{\iota+1} - t_\iota$, $0 \leq \iota < N$, is *infinitesimal*, *i.e.*, "very small".[9] A *time series* $X(t)$ is a function $X : \mathfrak{T} \to \mathbb{R}$.

The *Lebesgue measure* on $\mathfrak{T}$ is the function ℓ defined on $\mathfrak{T}\backslash\{1\}$ by $\ell(t_i) = t_{i+1} - t_i$. The measure of any interval $[c, d] \subset \mathfrak{T}$, $c \leq d$, is its length $d - c$. The *integral* over $[c, d]$ of the time series $X(t)$ is the sum

$$\int_{[c,d]} X d\tau = \sum_{t \in [c,d]} X(t)\ell(t)$$

X is said to be *S-integrable* if, and only if, for any interval $[c, d]$ the integral $\int_{[c,d]} |X| d\tau$ is *limited*, *i.e.*, not infinitely large, and, if $d - c$ is infinitesimal, also infinitesimal.

X is *S-continuous* at $t_\iota \in \mathfrak{T}$ if, and only if, $f(t_\iota) \simeq f(\tau)$ when $t_\iota \simeq \tau$.[10] X is said to be *almost continuous* if, and only if, it is S-continuous on $\mathfrak{T} \setminus R$, where R is a *rare* subset.[11] X is *Lebesgue integrable* if, and only if, it is S-integrable and almost continuous.

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[6] The connection between time series and control has been investigated quite a lot in the literature (see, *e.g.*, [2]).

[7] Note that those two references [28, 29] are studying Granger-causality for a better understanding of some questions stemming from neurosciences.

[8] See [12] for a more thorough introduction on this theorem, and on nonstandard analysis. Many more citations are also given.

[9] See, *e.g.*, [4, 5] for basics in nonstandard analysis.

[10] $a \simeq b$ means that $a - b$ is infinitesimal.

[11] The set R is said to be *rare* [3] if, for any standard real number $\alpha > 0$, there exists an internal set $A \supset R$ such that $m(A) \leq \alpha$.

A time series $\mathcal{X} : \mathfrak{T} \to \mathbb{R}$ is said to be *quickly fluctuating*, or *oscillating*, if, and only if, it is S-integrable and $\int_A \mathcal{X} d\tau$ is infinitesimal for any *quadrable* subset.[12]

Let $X : \mathfrak{T} \to \mathbb{R}$ be a S-integrable time series. Then, according to the Cartier-Perrin theorem [3],[13] the additive decomposition

$$\boxed{X(t) = E(X)(t) + X_{\text{fluctuat}}(t)} \tag{1}$$

holds where

- the *mean* $E(X)(t)$ is Lebesgue integrable,
- $X_{\text{fluctuat}}(t)$ is quickly fluctuating.

The decomposition (1) is unique up to an additive infinitesimal quantity.

Remark 3. The notion of quick fluctuations has been employed since [10] as a new approach to noise in automatic control and signal processing. Let us emphasize that this setting was successfully utilized for obtaining powerful estimation and identification techniques (see, *e.g.*, [18–21, 42], and the references therein). See [11] for another advances in signal processing, which are based on nonstandard analysis.

2.2. **Variances and covariances**

2.2.1. *Squares and products*

Take two S-integrable time series $X(t)$, $Y(t)$, such that their squares and the squares of $E(X)(t)$ and $E(Y)(t)$ are also S-integrable. The Cauchy-Schwarz inequality shows that the products

- $X(t)Y(t)$, $E(X)(t)E(Y)(t)$,
- $E(X)(t)Y_{\text{fluctuation}}(t)$, $X_{\text{fluctuation}}(t)E(Y)(t)$,
- $X_{\text{fluctuation}}(t)Y_{\text{fluctuation}}(t)$

are all S-integrable.

2.2.2. *Differentiability*

Assume moreover that $E(X)(t)$ and $E(Y)(t)$ are *differentiable* in the following sense: there exist two Lebesgue integrable time series $f, g : \mathfrak{T} \to \mathbb{R}$, such that, for any $t \in \mathfrak{T}$, with the possible exception of a limited number of values of t, $E(X)(t) = E(X)(0) + \int_0^t f(\tau)d\tau$, $E(Y)(t) = E(Y)(0) + \int_0^t g(\tau)d\tau$. Integrating by parts shows that the products $E(X)(t)Y_{\text{fluctuation}}(t)$ and $X_{\text{fluctuation}}(t)E(Y)(t)$ are quickly fluctuating [10].

Remark 4. Let us emphasize that the product

$$X_{\text{fluctuation}}(t)Y_{\text{fluctuation}}(t)$$

is not necessarily quickly fluctuating. This most easily verified by setting $X_{\text{fluctuation}}(t) = \pm 1$, and $Y_{\text{fluctuation}}(t) = X_{\text{fluctuation}}(t)$. Then

$$X_{\text{fluctuation}}(t)Y_{\text{fluctuation}}(t) = (X_{\text{fluctuation}}(t))^2 = 1$$

2.2.3. *Definitions*

(1) The *covariance* of two time series $X(t)$ and $Y(t)$ is

$$\begin{aligned} \operatorname{cov}(XY)(t) &= E\left((X - E(X))(Y - E(Y))\right)(t) \\ &\simeq E(XY)(t) - E(X)(t) \times E(Y)(t) \end{aligned}$$

[12]A set is *quadrable* [3] if its boundary is rare.

[13]The presentation in the article by Lobry and Sari [33] is less demanding. We highly recommend it. Note that it also includes a fruitful discussion on nonstandard analysis.

(2) The *variance* of the time series $X(t)$ is

$$\begin{aligned} \operatorname{var}(X)(t) &= E\left((X - E(X))^2\right)(t) \\ &\simeq E(X^2)(t) - (E(X)(t))^2 \end{aligned}$$

(3) The *volatility* of $X(t)$ is the corresponding standard deviation

$$\boxed{\operatorname{vol}(X)(t) = \sqrt{\operatorname{var}(X)(t)}} \tag{2}$$

The volatility of a quite arbitrary time series seems to be precisely defined here for the first time.

2.3. **Returns**

2.3.1. *Definition*

Assume from now on that, for any $t \in \mathfrak{T}$,

$$0 < m < X(t) < M$$

where m, M are *appreciable*.

Remark 5. This is a realistic assumption if $X(t)$ is the price of some financial asset. If $X(t)$ is a temperature, express it in Kelvin degrees, for instance.

The *logarithmic return*, or *log-return*, of X with respect to some limited time interval $\Delta T > 0$ is the time series $R_{\Delta T}$ defined by

$$R_{\Delta T}(X)(t) = \ln\left(\frac{X(t)}{X(t - \Delta T)}\right) = \ln X(t) - \ln X(t - \Delta T)$$

From $\frac{X(t)}{X(t-\Delta T)} = 1 + \frac{X(t)-X(t-\Delta T)}{X(t-\Delta T)}$, we know that

$$R_{\Delta T}(X)(t) \simeq \frac{X(t) - X(t - \Delta T)}{X(t - \Delta T)} \tag{3}$$

if $X(t) - X(t - \Delta T)$ is infinitesimal. The right handside of Equation (3) is the *arithmetic* return.

The *normalized* logarithmic return is

$$r_{\Delta T}(X)(t) = \frac{R_{\Delta T}(t)}{\Delta T} \tag{4}$$

2.3.2. *Mean*

Replace $X : \mathfrak{T} \to \mathbb{R}$ by

$$\ln X : \mathfrak{T} \to \mathbb{R}, \quad t \mapsto \ln (X(t))$$

where the logarithms of the prices are taken into account. Apply the Cartier-Perrin theorem to $\ln X$. The *mean*, or *average*, of $r_{\Delta T}(t)$ given by Equation (4) is

$$\boxed{\bar{r}_{\Delta T}(X)(t) = \frac{E(\ln X)(t) - E(\ln X)(t - \Delta T)}{\Delta T}} \tag{5}$$

As a matter of fact $r_{\Delta T}(X)$ and $\bar{r}_{\Delta T}(X)$ are related by

$$r_{\Delta T}(X)(t) = \bar{r}_{\Delta T}(X)(t) + \text{quick fluctuations}$$

Assume that $E(X)$ and $E(\ln X)$ are differentiable according to Section 2.2.2. Call the derivative of $E(\ln X)$ the *normalized mean logarithmic instantaneous return* and write

$$\boxed{\bar{r}(X)(t) = \frac{d}{dt}E(\ln X)(t)} \tag{6}$$

Note that $E(\ln X)(t) \simeq \ln\left(E(X)(t)\right)$ if in Equation (1) $X_{\text{fluctuation}}(t) \simeq 0$. Then $\bar{r}(X)(t) \simeq \frac{\frac{d}{dt}E(X)(t)}{E(X)(t)}$.

2.3.3. *Volatility*

Formulae (2), (4), (5), (6) yield the following mathematical definition of the *volatility* of the time series X when computed via its retun:

$$\boxed{\mathbf{vol}_{\Delta T}(X)(t) = \sqrt{E(r_{\Delta T} - \bar{r}_{\Delta T})^2(t)}} \tag{7}$$

It yields

$$\boxed{\mathbf{vol}_{\Delta T}(X)(t) \simeq \sqrt{E(r^2_{\Delta T})(t) - (\bar{r}_{\Delta T}(t))^2}}$$

3. Beta

It is well known that the coefficient β was introduced in financial engineering for studying some types of *risks*. The presentation below is inspired by [15].

3.1. **Arithmetical average**

Assume that $X : \mathfrak{T} \to \mathbb{R}$ is S-integrable. Take a quadrable set $A \subseteq \mathfrak{T}$ such that $\int_A d\tau$ is appreciable. The *arithmetical average* of X on A, which is written $\mathrm{AV}_A(X)$, is defined by

$$\boxed{\mathrm{AV}_A(X) = \frac{\int_A X d\tau}{\int_A d\tau}}$$

It follows at once from Equation (1) that the difference between $\mathrm{AV}_A(X)$ and $\mathrm{AV}_A(E(X))$ is infinitesimal, *i.e.*,

$$\mathrm{AV}_A(X) \simeq \mathrm{AV}_A(E(X))$$

In practice, A is a time interval $[t-L, t]$, with an appreciable length L. Set, if $t \geq L$,

$$\boxed{\overline{X}(L,t) = \mathrm{AV}_{[t-L,t]}(X) = \frac{\int_{t-L}^t X d\tau}{L} \simeq \frac{\int_{t-L}^t E(X) d\tau}{L}}$$

3.2. **A formula for betas**

Take two

- S-integrable time series $X, Y : \mathfrak{T} \to \mathbb{R}$,
- quantities $L_X, L_Y > 0$.

If $t > \sup(L_X, L_Y)$ and if $\overline{Y}(L_Y, t)$ is appreciable, set

$$\beta^{X,L_X}_{Y,L_Y}(t) = \frac{\overline{X}(L_X,t)}{\overline{Y}(L_Y,t)}$$

If $L_X = L_Y = L$, set

$$\boxed{\beta_Y^X(L,t) = \beta_{Y,L}^{X,L}(t) = \frac{\overline{X}(L,t)}{\overline{Y}(L,t)}} \tag{8}$$

The *relation*, or *influence*, between X and Y has been already defined in Section 1.3. It depends of course on the numerical values of $\beta_Y^X(L,t)$.

4. Numerical experiments

The academic time series, who are extracted from [29], *i.e.*, a paper on neurosciences, are given by closed form expressions. There is no room here for studying data from real life.

Remark 6. All the βs in this Section are computed by taking the returns of the time series.

4.1. Case 1

Figure 1 displays the two time series

$$\begin{cases} y_1(t) = \frac{t}{50} \\ y_2(t) = \sin(\frac{t^2}{200}) + 3\sin(\frac{t}{10}) \end{cases}$$

As shown by Figure 2 there is no clear-cut relation after some time, *i.e.*, $t \simeq 80$, with a short time lapse $L = 0.1$ in Equation (8). This is explained of course by the term $\sin(\frac{t^2}{200})$. If the time lapse L becomes larger, *i.e.*, $L = 10s$, a relation may be read on Figure 3, since the influence of $\sin(\frac{t^2}{200})$ is reduced.

4.2. Case 2

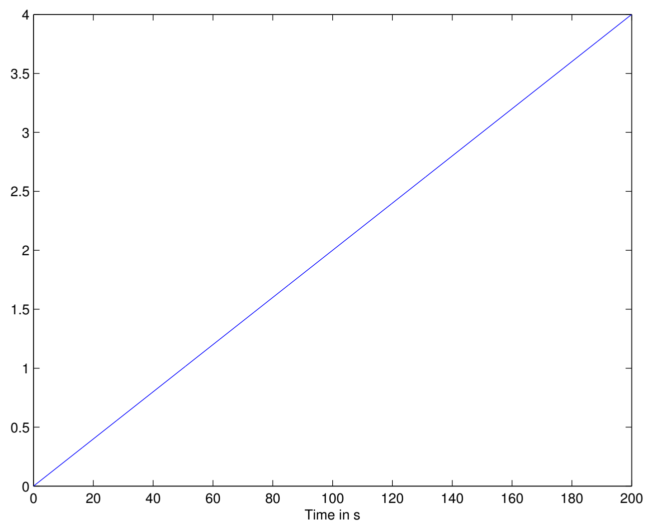
The five time series in Figure 4 are borrowed from [29]:

$$\begin{cases} x_1(t) = & .95\sqrt{2}x_1(t-1) - 0.9025x_1(t-2) + \epsilon_1(t) \\ & +a_1\epsilon_6(t) + b_1\epsilon_7(t) + c_1\epsilon_7(t-2) \\ x_2(t) = & .5x_1(t-2) + \epsilon_2(t) + a_2\epsilon_6(t) + b_2\epsilon_7(t-1) + c_2\epsilon_7(t-2) \\ x_3(t) = & -.4x_1(t-3) + \epsilon_3(t) + a_3\epsilon_6(t) + b_3\epsilon_7(t-1) + c_3\epsilon_7(t-2) \\ x_4(t) = & -.5x_1(t-2) + .25\sqrt{2}x_4(t-1) + .25\sqrt{2}x_5(t-1) + \epsilon_4(t) \\ & +a_4\epsilon_6(t) + b_4\epsilon_7(t-1) + c_4\epsilon_7(t-2) \\ x_5(t) = & -.25\sqrt{2}x_4(t-1) + .25\sqrt{2}x_5(t-1) + \epsilon_5(t) \\ & +a_5\epsilon_6(t) + b_5\epsilon_7(t-1) + c_5\epsilon_7(t-2) \end{cases} \tag{9}$$

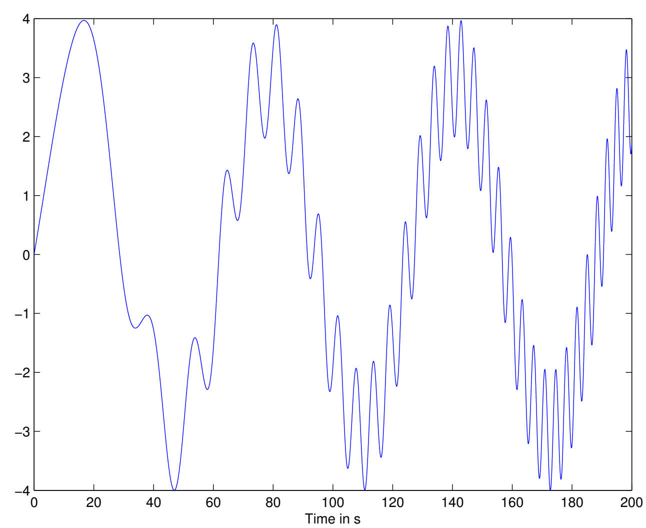
where

- $\epsilon_i(t)$, $i = 1, \cdots, 7$, are zero-mean uncorrelated processes with identical variances;
- the coefficients a_i, which represent exogenous inputs, are randomly chosen between 0 and 1;
- the terms $b_i\epsilon_7(t-1) + c_i\epsilon_7(t-2)$, $b_i = 2$, $c_i = 5$, $i = 1, 2, \cdots, 7$, represent the influence of latent variables.

Figure 5 displays the various β_j^i, $i, j = 1, \ldots, 7$, $i \neq j$, with a window length equal to $200T_e$, where T_e is the sampling time. The trends of those quantities are shown in Figure 6, which displays also the corresponding $25Te$ forecasts. Those numerical results, which are of good quality, give a clear-cut interpretation of the relations between the various time series. The trends are also presented in Figure 6. Figures 7 and 8 present respectively the corresponding volatilities of x_i's and the associated βs. Here again the relations may be clearly deduced.



(a) $y_1(t)$



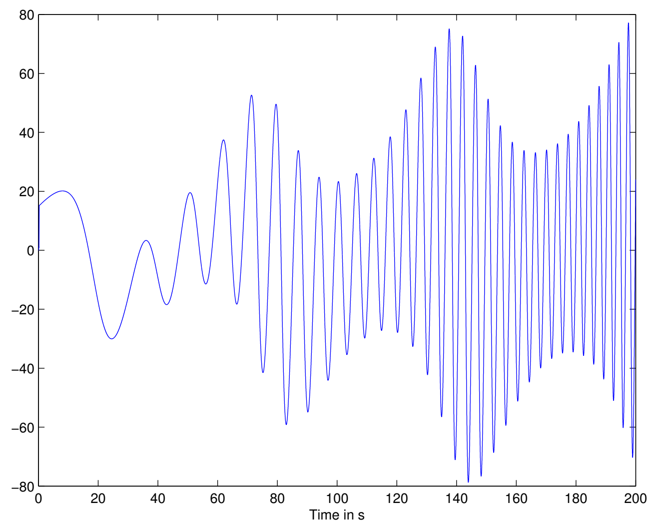
(b) $y_2(t)$

FIGURE 1. Time evolution of signals

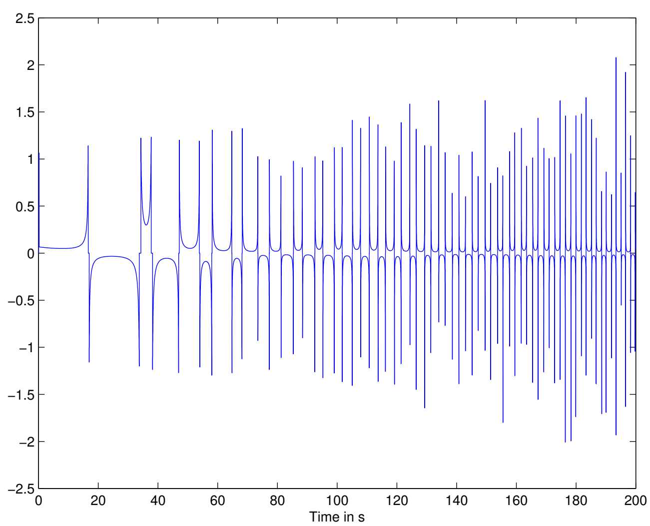
5. Conclusion

Let us stress again that this new setting for causality between time series, which has been outlined here, does not need any complex deterministic or probabilistic mathematical modeling.[14] It seems moreover to be rather straightforward to implement. Its interest, which is obviously connected to some questions about big data, will be hopefully soon confirmed via concrete case-studies, like, for instance, meteorology, where our approach to time series begins to be employed [31].

[14] See also the "epistemological" comments in [12, 17].



(a) $\beta_{y_2}^{y_1}$



(b) $\beta_{y_1}^{y_2}$

FIGURE 2. Betas computed on a short time interval $L = 0.1s$

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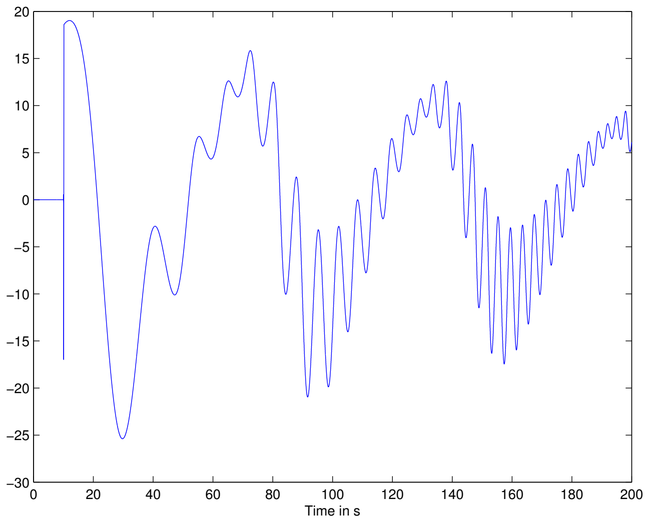
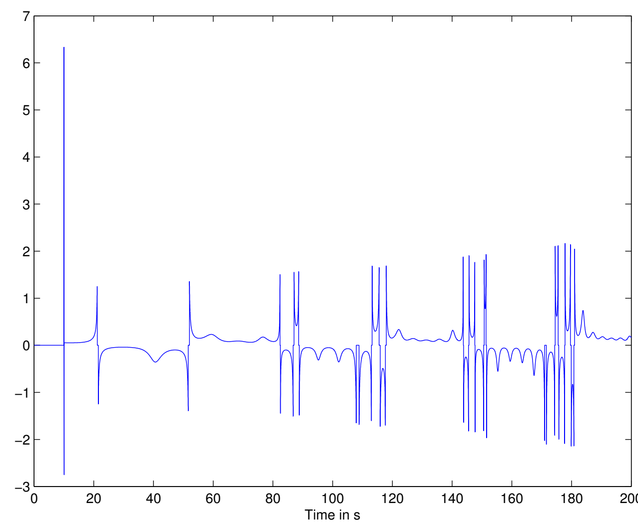


(a) $\beta_{y_2}^{y_1}$



(b) $\beta_{y_1}^{y_2}$

Figure 3. Betas computed on a long time interval $L = 10s$

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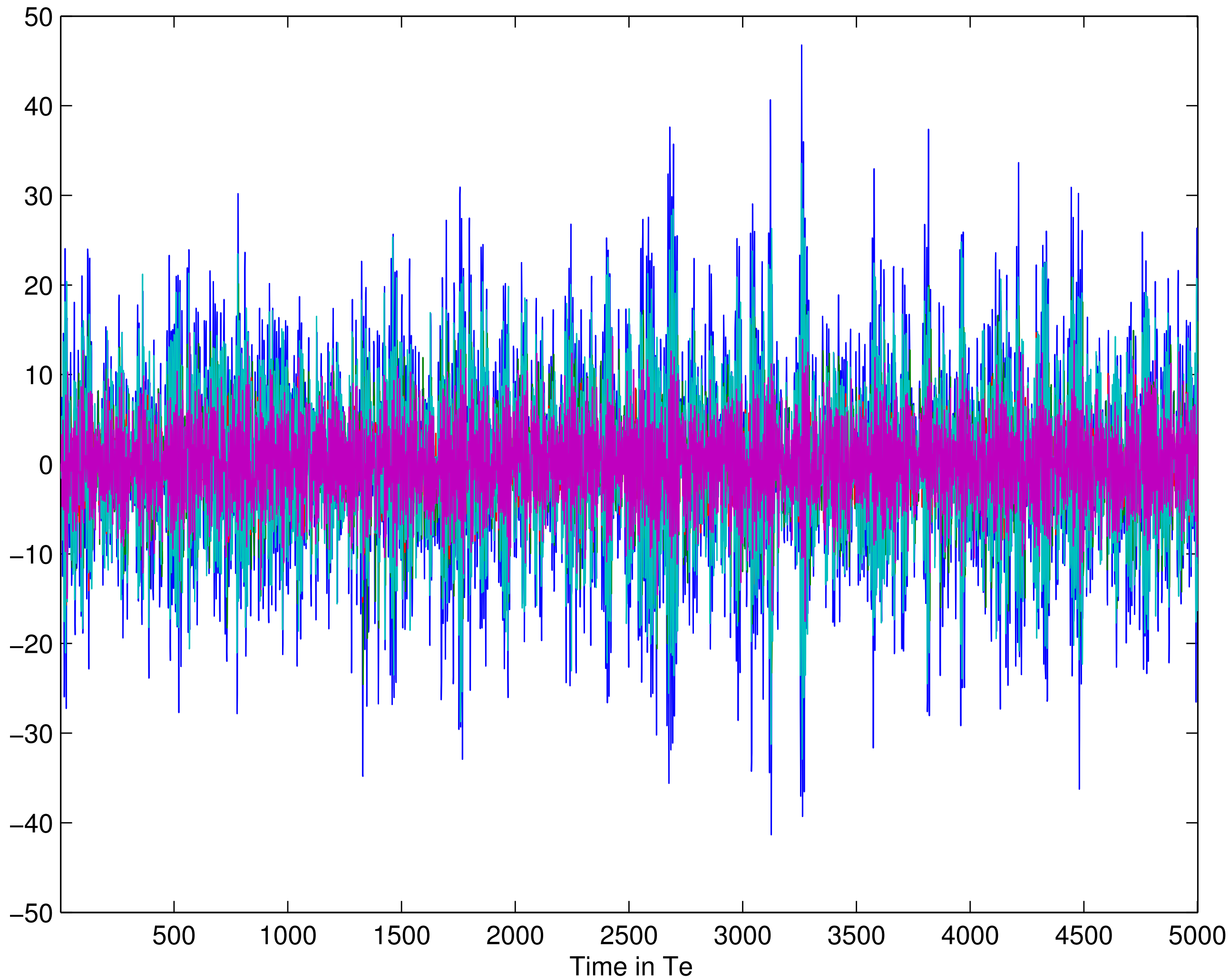

FIGURE 4. Time evolution of x_i, $i = 1, 2, \cdots, 7$

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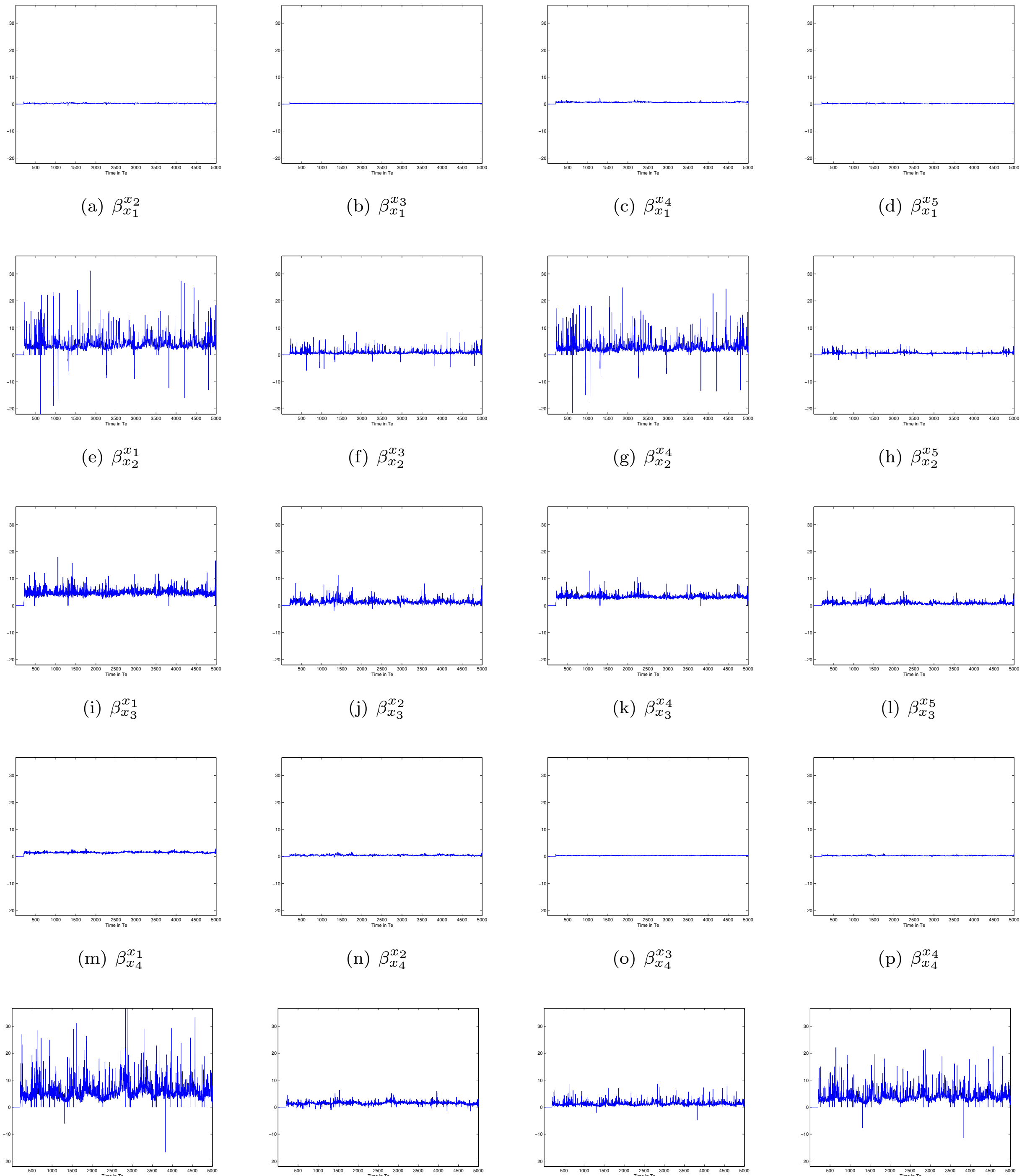


(a) $\beta_{x_1}^{x_2}$ (b) $\beta_{x_1}^{x_3}$ (c) $\beta_{x_1}^{x_4}$ (d) $\beta_{x_1}^{x_5}$

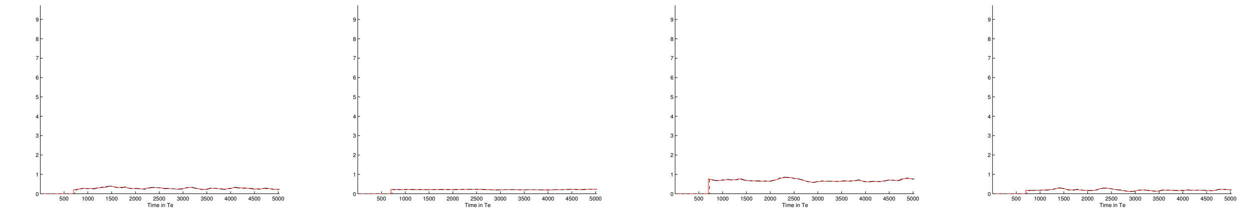
(e) $\beta_{x_2}^{x_1}$ (f) $\beta_{x_2}^{x_3}$ (g) $\beta_{x_2}^{x_4}$ (h) $\beta_{x_2}^{x_5}$

(i) $\beta_{x_3}^{x_1}$ (j) $\beta_{x_3}^{x_2}$ (k) $\beta_{x_3}^{x_4}$ (l) $\beta_{x_3}^{x_5}$

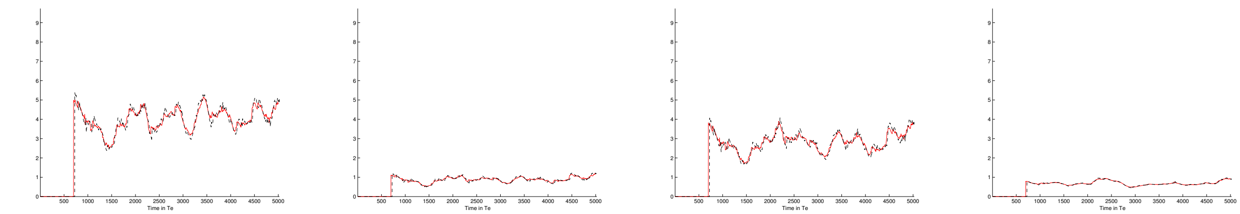
(m) $\beta_{x_4}^{x_1}$ (n) $\beta_{x_4}^{x_2}$ (o) $\beta_{x_4}^{x_3}$ (p) $\beta_{x_4}^{x_4}$

(q) $\beta_{x_5}^{x_1}$ (r) $\beta_{x_5}^{x_2}$ (s) $\beta_{x_5}^{x_3}$ (t) $\beta_{x_5}^{x_4}$

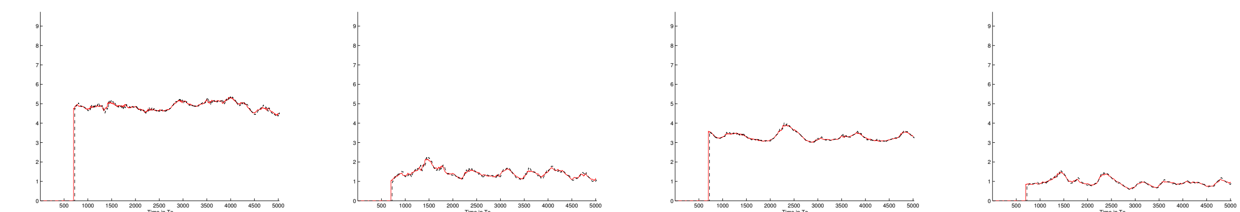
FIGURE 5. Betas for all combinations of x_i, $i = 1, 2, \cdots, 7$



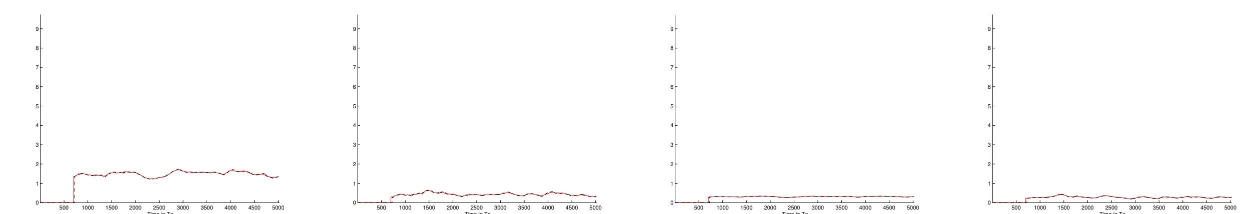
(a) Trend of $\beta_{x_1}^{x_2}$ (red $-$) and $25Te$ forecast (black $--$)
(b) Trend of $\beta_{x_1}^{x_3}$ (red $-$) and $25Te$ forecast (black $--$)
(c) Trend of $\beta_{x_1}^{x_4}$ (red $-$) and $25Te$ forecast (black $--$)
(d) Trend of $\beta_{x_1}^{x_5}$ (red $-$) and $25Te$ forecast (black $--$)



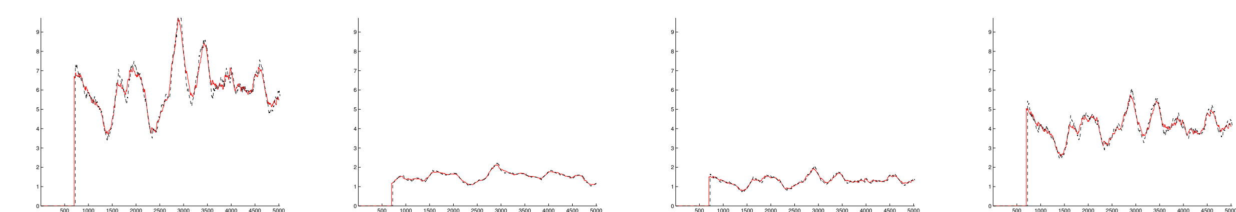
(e) Trend of $\beta_{x_2}^{x_1}$ (red $-$) and $25Te$ forecast (black $--$)
(f) Trend of $\beta_{x_2}^{x_3}$ (red $-$) and $25Te$ forecast (black $--$)
(g) Trend of $\beta_{x_2}^{x_4}$ (red $-$) and $25Te$ forecast (black $--$)
(h) Trend of $\beta_{x_2}^{x_5}$ (red $-$) and $25Te$ forecast (black $--$)



(i) Trend of $\beta_{x_3}^{x_1}$ (red $-$) and $25Te$ forecast (black $--$)
(j) Trend of $\beta_{x_3}^{x_2}$ (red $-$) and $25Te$ forecast (black $--$)
(k) Trend of $\beta_{x_3}^{x_4}$ (red $-$) and $25Te$ forecast (black $--$)
(l) Trend of $\beta_{x_3}^{x_5}$ (red $-$) and $25Te$ forecast (black $--$)



(m) Trend of $\beta_{x_4}^{x_1}$ (red $-$) and $25Te$ forecast (black $--$)
(n) Trend of $\beta_{x_4}^{x_2}$ (red $-$) and $25Te$ forecast (black $--$)
(o) Trend of $\beta_{x_4}^{x_3}$ (red $-$) and $25Te$ forecast (black $--$)
(p) Trend of $\beta_{x_4}^{x_4}$ (red $-$) and $25Te$ forecast (black $--$)



(q) Trend of $\beta_{x_5}^{x_1}$ (red $-$) and $25Te$ forecast (black $--$)
(r) Trend of $\beta_{x_5}^{x_2}$ (red $-$) and $25Te$ forecast (black $--$)
(s) Trend of $\beta_{x_5}^{x_3}$ (red $-$) and $25Te$ forecast (black $--$)
(t) Trend of $\beta_{x_5}^{x_4}$ (red $-$) and $25Te$ forecast (black $--$)

Figure 6. Forecasts of the βs

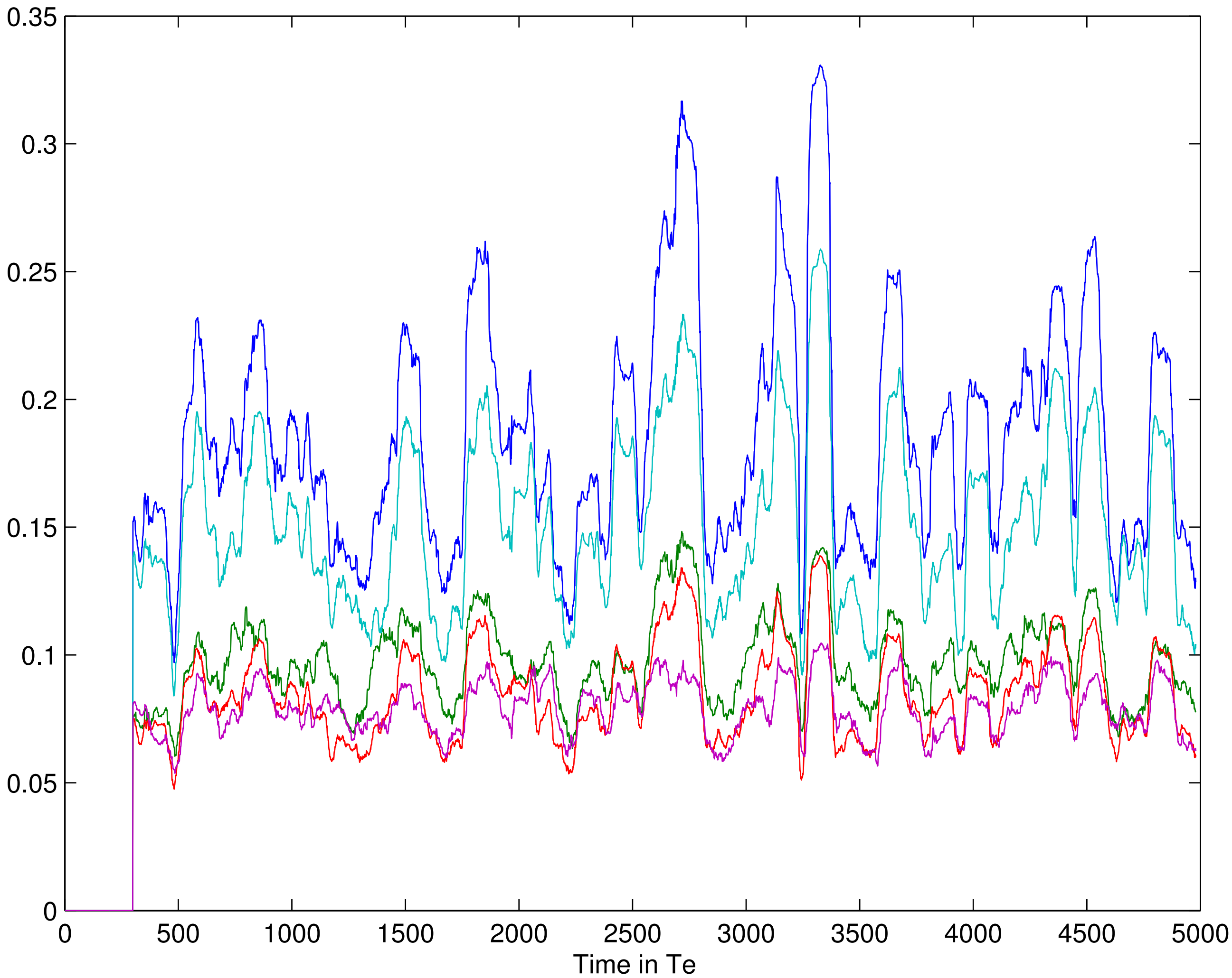


FIGURE 7. Time evolution of $\mathbf{vol}_{10Te}(x_i)(t)$, $i = 1, 2, \cdots, 7$

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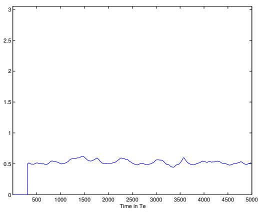
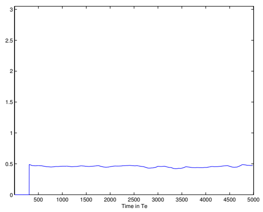
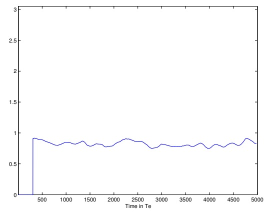


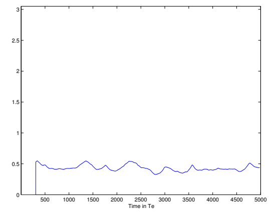
(a) $\beta_{x_1}^{x_2}$



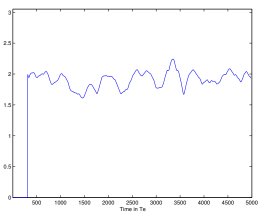
(b) $\beta_{x_1}^{x_3}$



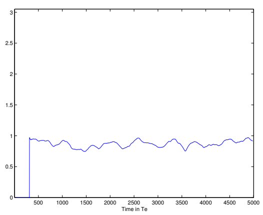
(c) $\beta_{x_1}^{x_4}$



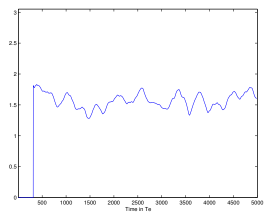
(d) $\beta_{x_1}^{x_5}$



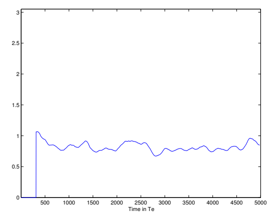
(e) $\beta_{x_2}^{x_1}$



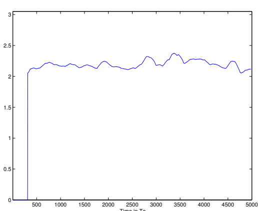
(f) $\beta_{x_2}^{x_3}$



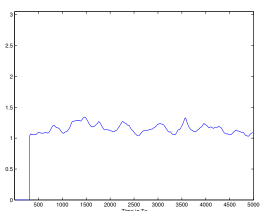
(g) $\beta_{x_2}^{x_4}$



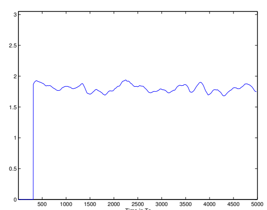
(h) $\beta_{x_2}^{x_5}$



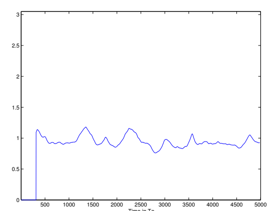
(i) $\beta_{x_3}^{x_1}$



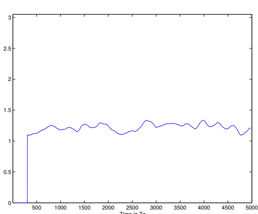
(j) $\beta_{x_3}^{x_2}$



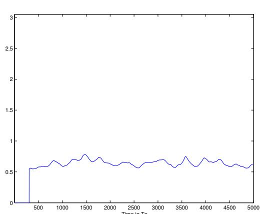
(k) $\beta_{x_3}^{x_4}$



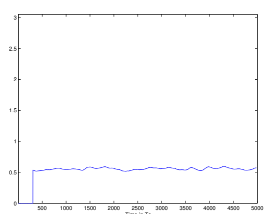
(l) $\beta_{x_3}^{x_5}$



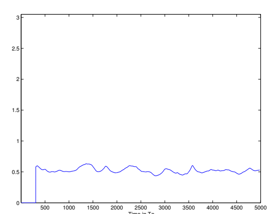
(m) $\beta_{x_4}^{x_1}$



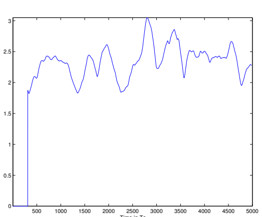
(n) $\beta_{x_4}^{x_2}$



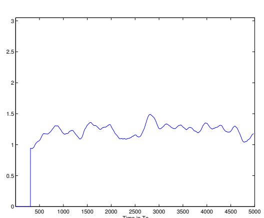
(o) $\beta_{x_4}^{x_3}$



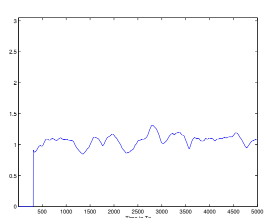
(p) $\beta_{x_4}^{x_4}$



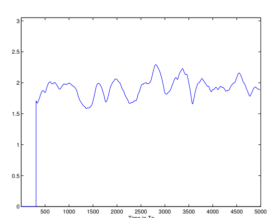
(q) $\beta_{x_5}^{x_1}$



(r) $\beta_{x_5}^{x_2}$



(s) $\beta_{x_5}^{x_3}$



(t) $\beta_{x_5}^{x_4}$

FIGURE 8. βs for all combinations of $\mathbf{vol}_{10Te}(x_i)(t)$, $i = 1, 2, \cdots, 7$

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