Propagating Pattern Selection and Causality Reconsidered

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Pattern selection, occurring when a nonuniform state of a nonlinear dissipative system propagates into an initially unstable, homogeneous basic state is reconsidered by application of the causality principle. In particular, the nonlinear marginal stability criterion that determines the selection of a nonlinear front solution is replaced by an exact general necessary condition that has never been considered before. The demonstration is based on the causal signaling problem derived in the context of plasma physics.

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From macro- to microscales, the Universe presents a striking mixture of order within disorder and of disorder within order. This leads to a great variety of patterns from solar granulation [1] to sunflower Fibonacci spirals [2], and from life cycles (population dynamics) [3] to heart attack fibrillations [4]. The ultimate goal of the theoretical analyses in pattern forming systems is "to understand whether, or under what circumstances, pattern formation (...) is an intrinsic property of the systems themselves or, perhaps, depends sensitively on initial configurations, boundaries, or externally imposed perturbations" [5]. In the past decade, attention has been focused on elementary models exhibiting pattern formation and that are theoretically and numerically tractable [6]. Insights gained on these toy models have then been extrapolated or extended to real systems such as Rayleigh-Bénard convection [7], cellular or dendritic solidification fronts [8], cellular flame fronts [9], liquid crystals [10], or solar dynamo waves [11].

Of particular interest is the pattern propagation, and great attention [12,13] has been paid to situations where a stable nonuniform state of a nonlinear dissipative system, generated from a localized initial perturbation, propagates into an initially unstable or metastable homogeneous infinite domain. The interface separating the stable nonuniform state from the unstable homogeneous state is a front, which moves at a constant velocity in several experiments [10,14], and an important part of the theoretical work about front propagation has been devoted to the understanding of the selection principle which determines the velocity of the front and the typical wavelength of the nonlinear stable state behind. The front propagation may be of two types [12,13] (pulled or pushed regime) schematically depending on whether the state into which the front propagates is unstable or metastable. To explain this selection, a linear (pulled regime) and a nonlinear (pushed regime) marginal stability criterion have been proposed [13] and compare favorably with the experiments in both cases [10,14], but no systematic derivation of these criteria has been given. The reader is referred to the literature for a detailed discussion of these criteria, and we will present here the linear selection criterion developed by Dee and Langer [5] and rephrased by Huerre and Monkewitz [15]. For simplicity, let us consider a one-dimensional problem with x, the spatial coordinate. For an unstable basic state, the linear Green function defines the linear propagation of an initial impulse perturbation. The linear response to any initially localized perturbation will be obtained by convolution of the Fourier transform of the perturbation with the Green function. At long time, the response will be dictated by the long time dynamics of the Green function. In particular, it will be exponentially decaying outside a wedge (or several wedges) limited by spatiotemporal rays $x/t = v_l$ (t being the time since the impulse has been applied) such that

$$\frac{d\omega}{dk} = v_l\,, \tag{1}$$

$$\mathrm{Im}(\omega - v_l k) = 0\,, \tag{2}$$

with ω being the complex frequency and k the complex wave number for which the plane wave $\exp[i(kx - \omega t)]$ is a solution of the linearized equation of motion; i.e., ω and k are solutions of the linear dispersion relation $D(\omega, k) = 0$. Conditions (1) and (2) correspond to the classical marginal absolute instability criterion. In fact, from the works of Briggs and Bers [16] on plasma waves, we know that these conditions are not sufficient and an extra "pinching" condition, that will be detailed later, has to be fulfilled.

The nonlinear response will be obtained by letting the wave packet saturate, and it is legitimate to imagine that saturation will occur inside the unstable wedge where the amplitude would be exponentially growing if saturating nonlinearities were absent. The fronts bounding the saturated state are then the edges defined by the velocities v_l satisfying Eqs. (1) and (2). This linear selection principle was first derived by Kolmogorov *et al.* [12] and is fully equivalent to the marginal stability criterion. It may be rephrased by saying that the front which is selected from the linear criterion, also referred to as the Kolmogorov front or the pulled front, is such that, in the frame moving with the front, the basic state is at the absolute instability threshold. If the front were going slower than v_l, it would be unstable since any infinitely small perturbation would

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develop and propagate ahead of the front. Fronts moving faster than v_l are possible since perturbations ahead of the front are advected toward the front and are eventually absorbed in the saturated region. The linear selection criterion is therefore not sufficient, and among these fast fronts one must determine which is dynamically selected. A nonlinear selection principle has been proposed to that aim [13], and the nonlinearly selected front is referred to as the nonlinear front or the pushed front.

The existence and selection of a pushed front is obvious when considering a metastable basic state. In this case, the linear analysis does not enable us to predict the propagation of the bifurcated domain since the linear Green function is damped at any location in any moving frame. The pattern propagation front is therefore nonlinearly determined by solving in any frame moving at velocity v the problem of existence of a purely time-periodic solution at the unknown frequency ω, asymptotic to a saturated plane wave at $-\infty$ and to the basic state at $+\infty$. In this metastable case, when several such fronts exist and move at different velocities, the selected one is merely the fastest. But the existence of such a nonlinear front is not limited to the stable parameter range. At the linear instability threshold, front solutions start to exist for any propagation velocity and the selected one may be determined by "continuity" [17] with the linearly unstable range. Nonlinearly selected fronts may be found not only in systems exhibiting a subcritical bifurcation but also for systems undergoing a purely supercritical bifurcation [13,18]. In the latter case, the selection cannot be determined by a continuity argument. The selection principle (extended marginal stability criterion) conjectured and tested numerically in Ref. [13] may be stated as "the front is nonlinearly selected (pushed) when it is simultaneously faster and steeper (with the steepness defined as the spatial decreasing at $x \to +\infty$) than the corresponding linearly selected (pulled) front" [13].

The purpose of the present paper is to give an exact nonlinear selection criterion based on the causality principle. For a generalized complex Ginzburg-Landau equation, the new criteria enable us to recover former results whereas, for a nonlinear problem involving a higher order linear dispersion relation, it will predict a selection that may differ from the marginal stability principle. We illustrate the new criterion on the real subcritical Ginzburg-Landau equation describing a pitchfork bifurcation in an extended system for which front selection has been exactly determined [13]:

$$\partial_t A = \partial_{xx} A + \mu A + (2\alpha - 1)A^3 - \alpha A^5, \qquad (3)$$

with $A(x,t)$ the order parameter, μ the bifurcation parameter which will be taken positive in the following, and α a coefficient which equals to unity, except when otherwise specified for the purpose of illustration. Figure 1 shows for $\mu > 0$ the different phase diagrams $(A, dA/dx)$ for steady front solutions in the frame moving at velocity v, i.e., fronts in the form $A(x - vt)$, where v is the constant asymptotic velocity, linking a saturated nonlinear state at $x = -\infty$ to an unstable state when $x = +\infty$. Front solutions are sketched as heteroclinic orbits which depart from the fixed point $A_2 \equiv [A = (1/2 + \sqrt{\mu + 1/4})^{1/2}, dA/dx = 0]$ and arrive at the fixed point $A_0 \equiv (A = 0, dA/dx = 0)$ along a specific eigendirection, the slope of which is linked to the steepness of the front at $x \to +\infty$ (a large slope indicates a steep front). For $v < 2\sqrt{\mu}$, the heteroclinic orbit spirals around A_0 which is a focus [Fig. 1(a)]. For $v > 2\sqrt{\mu}$ the heteroclinic orbit asymptotes generically to A_0 on the least stable eigendirection [Fig. 1(b)]. For $v = 2\sqrt{\mu}$, both stable eigendirections merge [Fig. 1(c)]. When $\mu < 3/4$, for discrete values of v, some heteroclinic orbits arrive at A_0 along the most stable eigendirection. The largest velocity v for which such an exceptional heteroclinic orbit exists [Fig. 1(d)] is given by

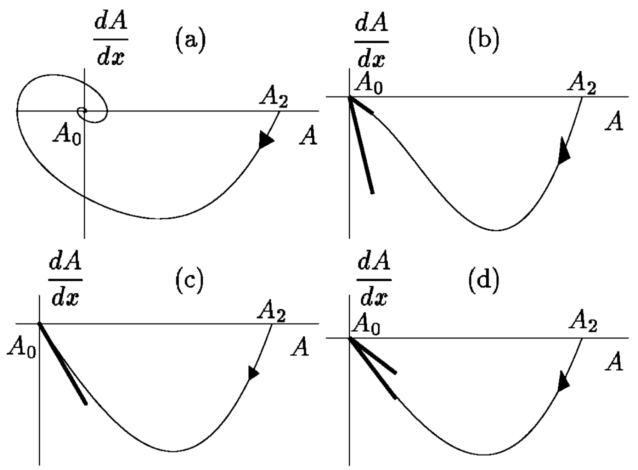

FIG. 1. Phase diagrams for front solutions with velocity v. The bold lines indicate the stable eigendirections of the fixed point A_0. (a) $v < 2\sqrt{\mu}$. (b) $v > 2\sqrt{\mu}$. (c) $v = 2\sqrt{\mu}$. Both stable eigendirections of A_0 have merged. (d) $v = v^{nl}(\mu)$ and $\mu < 3/4$.

$$v^{nl}(\mu) \equiv -\sqrt{3} + \frac{4}{\sqrt{3}}(1/2 + \sqrt{\mu + 1/4}). \qquad (4)$$

Following the above-mentioned conjecture, the front moves at the linearly selected velocity $v^l \equiv 2\sqrt{\mu}$ when $\mu > 3/4$, but at the nonlinearly selected velocity v^{nl} larger than v^l when $\mu < 3/4$.

When a higher dimensional equation is considered, the choice of the stable manifold to connect with at the origin may involve more than two eigendirections and the nonlinear selection principle should be extended and proven. In this case, the linear criterion is still valid and the selected velocity is the value v^l at which two eigenvalues of the evolution operator linearized around A_0 are equal, forcing the corresponding eigensubspaces to coalesce and recompose. In a frame moving slower than the linearly selected front, i.e., at velocity $v < v^l$, the instability of the basic state is linearly absolute, whereas, in a frame moving faster than v^l, the instability is convective. As argued in many places [12,13,18] a nonlinear front, excited from a localized initial condition, should be linearly stable and therefore should move faster than v^l. In other words, the instability of the basic state should be convective in the frame moving with a nonlinear front to avoid initial transient propagating ahead of the front. But, for a linearly unstable basic state, a front solution exists for any velocity $v > v^l$ and one should determine which one, if any, is dynamically correct, i.e., which one appears from initially localized perturbations.

To answer this question, we shall first consider the related problem of the linear response to a forcing at frequency ω, applied continuously at a fixed location $x = 0$, in a frame moving at velocity $v > v^l$. This "signaling" problem has been fully solved in plasma physics by Bers [16] and involves the subtle problem of causality. For a forcing which starts at time $t = 0$, we shall require that no response precedes the cause, i.e., that the response to the forcing is identically zero for $t < 0$. Bers [16], making use of spatial and temporal Laplace transform, has shown that the causality condition imposes the dispersion relation for linear plane waves $D(\omega, k)$ to be such that the temporal growth rates $\mathrm{Im}[\omega(k)]$ of temporal modes (homogeneous in space, i.e., k real) are bounded, let us say by $\omega_{i\max}$ [Fig. 2(a)]. A similar requirement does not hold for spatial modes (ω real) which may lie on both sides of the real k axis and spatial growth rates are not constrained to be bounded [Fig. 2(b)].

The formal symmetry between ω and k in the dispersion relation $D(\omega, k)$ is therefore broken by the causality which specifies that "information" propagates only toward positive time but may propagate in any direction in space. In particular, in the signaling problem, one should determine on which side of the source each spatial branch $k(\omega)$ propagates. When the basic state is stable, common sense tell us that the waves should be damped (evanescent) while propagating away from the source [Fig. 2(c1)] and that, therefore, branches k^+ with positive $\mathrm{Im}(k)$ propagate toward $+\infty$, whereas k^- waves with negative $\mathrm{Im}(k)$ propagate toward $-\infty$. To prove this conjecture and extend it to an unstable basic state, Bers does not refer to the concept of group velocity since the group velocity $d\omega/dk$ is in general complex, but to the causality principle taking the forcing source of the form $\delta(x)H(t)\exp(-i\omega t)$ with causality insured by the Heaviside function $H(t)$. Using contour closure to compute the inverse Laplace transform, he shows that the signaling problem has no solution when the instability is absolute since physically the initial transients are exponentially growing at the source. When the instability is convective, the initial transients are moving away from the source and the spatial branches separate between k^+- and k^--type depending on which side of the source they propagate [Fig. 2(c2)]. In the latter case, differentiation between k^+ and k^- branches does not rely on the sign of $\mathrm{Im}(k)$ since amplified waves are possible but on a contours deformation procedure which enables us to evaluate the Laplace transform of the response. Schematically a k^+ branch may be deformed continuously by increasing the imaginary part of ω to a branch lying entirely in the upper half of the complex k plane (Fig. 2(b2); see [16] for details). When the instability is absolute, the signaling problem is ill posed and k^+ and k^- branches cannot be differentiated as in the convective case. The causality principle cannot be applied, but this case is not interesting for front propagation since the pulled front is faster than any front in the reference frame of which the instability is absolute.

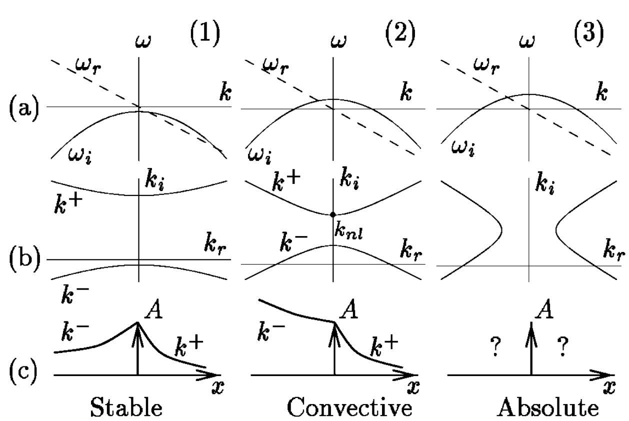

FIG. 2. (a) Temporal modes $\omega(k)$, k real; (b) spatial branches $k(\omega)$, ω real, in k plane; (c) response to forcing localized in space and harmonic in time. (1) Stable case; (2) convectively unstable case; (3) absolutely unstable case.

In a frame moving at a velocity larger than the linear velocity v_l, the instability is convective and a nonlinear front, issuing from a localized initial perturbation, stationary in that frame should emit a k^+ wave ahead of it, in the linear region where the amplitude is small enough for the linear theory to apply (a k^- linear wave propagates information from $+\infty$, i.e., requires a source at $+\infty$). This condition is necessary only for the front solution to be generated from an initially localized perturbation, and applies, in particular, to the case of pushed (or nonlinearly selected) fronts. In other words, a front moving at a velocity larger than v_l, the velocity of the pulled (or linearly selected) front, must connect the unstable state ahead of the nonlinear region to a k^+ linear branch in order to be causal. Clearly in this case, the pushed front will act as a wave maker in the frame moving with the front, radiating a k^+ linear wave ahead. These dynamics contrast with the pulled (linearly selected) front for which the wave maker is the edge of the linear Green function. Just behind the edge, the amplitude grows exponentially in space and saturates when nonlinear effects come into play. The pulled front is unambiguously causal since the velocity of the pulled front is determined by a saddle point of the dispersion relation [Eqs. (1) and (2)] of instability waves supplemented by a pinching condition between k^- and k^+ waves. As emphasized in [19], this pinching condition linked to causality implies that the sign condition $\mathrm{Im}[\partial^2\omega/\partial k^2] < 0$ be satisfied at the pinch point. To illustrate the difference between these causality considerations and the extended marginal stability criterion, let us consider the Nozaki-Bekki front [20], an analytic solution of Eq. (3) in the case of a supercritical bifurcation when $\alpha = 0$, which reads

$$A = \sqrt{\mu}/\{1 + \exp[\kappa_{\mathrm{NB}}(x - v_{\mathrm{NB}}t)]\}, \qquad (5)$$

with $v_{\mathrm{NB}} = 3\sqrt{\mu/2}$ and $\kappa_{\mathrm{NB}} = \sqrt{\mu}/2$. The pulled front shape cannot be determined analytically but it moves at the velocity $v^l = 2\sqrt{\mu}$ and its steepness equals $\kappa_l = \sqrt{\mu}$. The velocity v^l is smaller than v_{NB}, and the nonlinear selection criterion stipulates that the Nozaki-Bekki front will not be selected since its steepness κ_{NB} is smaller than κ_l. Hence, the selected front is the pulled front. In this case, the extended marginal stability criterion determines the right selected front. However, the causality condition would have determined here which front is selected without referring to the somehow intuitive idea of steepness of the front. The Nozaki-Bekki front indeed links a k^- linear branch when $x = +\infty$ and therefore is not a causal front. Our purpose has been illustrated on a very simple real amplitude equation for which the fronts have a zero frequency; however, the results still hold for complex and higher order amplitude equations. Furthermore, the same argument holds for the propagation of other types of solutions, for example, pulses which should connect a k^- wave at $x = -\infty$ to a k^+ wave at $x = +\infty$, although the case of pulses is less ambiguous since they propagate usually in a stable medium.

For a simple dispersion relation such as the one obtained from the linear Ginzburg-Landau equation, we have in the convective case the particular property that, for any ω, $\mathrm{Im}[k^+(\omega)] > \mathrm{Im}[k^-(\omega)]$. Our condition that dynamically accessible fronts should be asymptotic to a k^+ wave corresponds to the steepness conjecture in the nonlinear selection criterion [13]. But it may happen in more complex systems such as in the instability of a boundary layer over a membrane [21] that, for some ω and some particular spatial branches, $\mathrm{Im}[k^+(\omega)] < \mathrm{Im}[k^-(\omega)]$. The extended marginal stability criterion (nonlinear selection principle) applied to such a case will predict a noncausal front solution receiving energy from $+\infty$. The present criterion that a pushed front should be causal, and therefore asymptotic to a k^+ linear wave ahead of the front, would determine the correct selection of the front velocity. It constitutes a definite extension of the previous selection conjectures since it gives up the intuitive idea of steepness of the fronts. In the pulled regime, the causality considerations show that all fronts with velocity larger than v_l constitute a continuum of noncausal fronts and are not excitable by a localized perturbation.

We have therefore proposed a front selection criterion based upon the physical concept of causality. This new criterion merely reads "the selected front (if any) is the fastest causal front," and recovers the nonambiguous case of the marginal stability criterion when the fastest causal front coincides with the steepest.

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