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Abstract

This paper relies on a Ramsey model with money to offer a simple theory of secular stagnation. The permanent failure of the economy to produce at full capacity results from three features: (i) The combination of the zero lower bound on the nominal interest rate and of an inflation ceiling imposes a lower bound on the real interest rate; (ii) Some dynastic households have a high propensity to save, due to a preference for wealth; (iii) A downward wage rigidity breaks the deflationary spiral resulting from the lack of demand. In this framework, I derive the paradox of flexibility, of thrift, and of toil.

If the inflation ceiling cannot be raised, then the government needs to rely on fiscal policy to escape secular stagnation. However, a conventional fiscal stimulus is not an efficient response to a permanent liquidity trap, and can even be welfare reducing. The solution is instead to tax household wealth and to subsidize income from physical capital, through an investment subsidy or a reduction in the taxation of corporate income. This optimal policy is revenue neutral and implements the first-best allocation of resources. However, to avoid a jump in the price level upon implementation of the optimal policy, the government needs to redeem the money that had previously been supplied to finance public deficits.

Keywords: Liquidity trap, Monetary and fiscal policy, Secular stagnation

JEL Classification: E12, E31, E62, E63

1 Introduction

Keynes (1936) explained how an economy can be depressed due to a lack of demand. Indeed, if households' demand for consumption and firms' demand for investment are excessively low, then the economy fails to produce at full capacity. While the resulting

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depressions are usually seen as an extreme manifestation of a business cycle phenomenon, Hansen (1939) was the first to worry that it might be a permanent state of affairs. This is the "secular stagnation" hypothesis.

When aggregate demand is weak, inflation declines and the central bank responds by aggressively cutting the nominal interest rate such as to stimulate demand. However, a fundamental constraint on monetary policy is that the nominal interest rate cannot be negative. Indeed, no-one is willing to pay more than \$100 for a bond yielding \$100 in the future. In the mid-1990s, Japan was the first large industrialized country to hit the zero lower bound and to fall into the liquidity trap. Its subsequent history, with an economy mired in very low inflation and weak GDP growth, suggests that there is no mechanism through which an economy naturally recovers from a persistent lack of demand. Moreover, Japan is facing a policy conundrum as aggregate demand remains desperately weak despite a budget deficit of about 7% of GDP, a debt-to-GDP ratio of 230%, and substantial monetary accommodation.

Over the past few years, there has been growing concerns that the U.S. and the Eurozone might be in a similar situation as Japan. In particular, Summers (2014) has emphasized that the credit boom of 2003-2007 in the U.S. did not generate a corresponding economic boom, which suggests that the unsustainable demand for consumption of poor borrowers was hiding an already weak level of demand of rich lenders.

Despite the prominence of the secular stagnation hypothesis in the policy debate, there has been few attempts to model it explicitly. In this paper, I offer a simple theory of secular stagnation, on which I then rely for a careful policy analysis. The structure of the model is a Ramsey economy with money and flexible prices, to which I add three features.

First, I assume that the central bank imposes a ceiling on inflation. This, together with the zero lower bound on the nominal interest rate, generates a lower bound on the real interest rate. In most industrialized countries, central banks never allow inflation to exceed 2%, which prevents the real interest rate from ever falling below -2%.

Second, I assume that households derive utility from holding wealth. This can raise their propensity to save to such an extent that a negative real interest rate is necessary for the economy to produce at full capacity. Importantly, infinitely lived households should be interpreted as dynasties. Hence, bequest motives are important determinants of the saving behavior of households. In particular, we know that pure altruism alone cannot account for the observed patterns of bequests (Kopczuk 2009). Instead, parents seem to be commonly motivated by the "joy of giving". A "capitalist spirit" also induces many individuals to derive intrinsic utility from the accumulation of wealth, which is then passed on from generation to generation (Carroll 2000, Kopczuk 2007).[1] While I rely for

[1] Also, Piketty (2011) relies on a model with wealth in the utility to show that the inheritance-to-

simplicity on a representative household model, my results only require that a strictly positive mass of households has a preference for wealth. Indeed, a well known result in macroeconomics is that the real interest must eventually be determined by the behavior of the most patient households. This is highly relevant as the rise in inequality is often cited as a major cause of the recent decline in aggregate demand (Summers 2014). A preference for wealth seems a natural explanation for the concentration of wealth in the hands of a small number of households with a very low propensity to spend (Kumhof, Rancière and Winant 2015).

Finally, I impose a downward wage rigidity. If the inflation ceiling is sufficiently low and the preference for wealth sufficiently strong, then aggregate demand is smaller than aggregate supply. This reduces inflation, which raises the real interest rate, which further contracts aggregate demand. To have a stationary secular stagnation equilibrium, I therefore need to put a break on the deflationary spiral. This is achieved through the downward wage rigidity.

In the secular stagnation equilibrium, aggregate demand is depressed. This induces firms' labor demand to fall short of workers' labor supply. However, this inefficiency is primarily due to the lower bound on the real interest rate, not to the downward wage rigidity. Indeed, if wages are more flexible, then inflation is smaller, the real interest is higher, and the economy is even more depressed. Conversely, for a sufficiently high inflation ceiling, there always exists a frictionless steady state equilibrium where the economy produces at full capacity.[2]

The obvious policy response to secular stagnation is to raise the inflation ceiling. However, in most countries, this seems politically and institutionally out of reach. Hence, I explore alternative remedies. A simple policy analysis confirms that the usual set of tools used for macroeconomic stabilization are not adequate in the context of secular stagnation. Increasing the money supply through open market operations is useless in a liquidity trap. A helicopter drop of money that is sufficiently large to induce the economy to produce at full capacity is inconsistent with the inflation ceiling. A fiscal stimulus raises firms' demand for labor much more than workers' consumption level. Hence, it is likely to be welfare reducing, even though the government spending multiplier is typically well above 1.

However, with a wider set of policy instruments, the government can implement the first-best allocation of resources without raising the inflation ceiling. To understand this result, note that secular stagnation is fundamentally due to an excessively high real

income ratio is a rising function of $r - g$, consistently with the historical experience of France from 1820 to 2009.

[2] Interestingly, in the secular stagnation equilibrium, inflation is essentially determined by the downward wage rigidity. Hence, the real interest rate is typically well above its lower bound, which results in a very depressed level of aggregate demand.

interest rate. This creates two distortions. First, it raises the cost of capital. This reduces the demand for investment, resulting in depressed capital-labor ratio. Second, a high real interest rate reduces the demand for consumption. The first distortion should be addressed by subsidizing the income from physical capital. This can either be achieved through an investment subsidy or a reduction in the taxation of corporate income. The second distortion can be offset through a wealth tax, which reduces the effective real interest rate faced by consumers. This policy is revenue neutral and it exactly replicates the allocation of resources that would result from an increase in the inflation ceiling.

By inducing the economy to produce at full capacity, the optimal policy makes the price level proportional to the money supply. Hence, to prevent a jump in the price level upon implementation of the policy, the government must redeem the money supply that is not used for transactions. This suggests that relying on the printing press to finance some stimulus can eventually make the government reluctant to implement the optimal policy.

Related Literature. Secular stagnation is a stationary phenomenon. However, in typical models of the liquidity trap, the binding zero lower bound forces the real interest rate to be above households' discount rate (Krugman 1998, Eggertsson and Woodford 2003, Werning 2012). This induces households to adopt a rising path of consumption, which is inconsistent with stationarity. A model of secular stagnation must therefore reconcile an excessively high real interest rate, which depresses aggregate demand, with the existence of a stationary equilibrium.

Eggertsson and Mehrotra (2014) solved the problem by relying on an overlapping generation model without altruistic links. In their theory of secular stagnation, aggregate demand is either depressed because of a low rate of population growth or because the economy has been hit by a deleveraging shock.[3] Interestingly, to eliminate the output gap, the government can either increase its spending or redistribute income from lenders to borrowers.

Kocherlakota (2013) showed that, with overlapping generations or credit constraints, secular stagnation can result from a fall in the price of land. Caballero and Farhi (2015) relied on a perpetual youth model with risky assets to offer an alternative theory of secular stagnation where the lack of demand is due to a shortage of safe assets. To escape the depression, the government should either allow inflation to increase or raise the supply of public debt, possibly by buying private risky assets.

My contribution to this literature is to show that secular stagnation can be derived in a Ramsey economy where some households have a preference for wealth. Under a dynastic interpretation of infinitely lived households, my work shows that altruistic links across

[3] They also considered the possibility that demand is depressed due to a rise in inequality, where agents with low propensity to consume become wealthier, or to a fall in the price of investment goods.

generations are not inconsistent with the existence of a stationary secular stagnation equilibrium. On the contrary, the bequest motive seems to be a natural justification for the preference for wealth that delivers stationarity. The Ramsey framework is convenient for policy analysis. This allows me to characterize the optimal policy that implements the first-best allocation of resources without raising the inflation ceiling. Also, by introducing money into the economy, I can investigate the monetary financing of public deficits.

In a related contribution, Michaillat and Saez (2015) offer a model of the business cycle where self-employed households exchange their labor services under matching frictions. As prices and market tightness are not independently determined, they focus on an equilibrium where inflation is constant and where market tightness adjusts such as to equate aggregate demand to aggregate supply. Hence, business cycle fluctuations result in inefficient fluctuations in market tightness. To have a permanent liquidity trap, they also assume that households have a preference for wealth. Interestingly, as a lack of demand does not translate into lower inflation, they do not obtain the Keynesian paradoxes.

Another strand of the literature has emphasized the role of expectations. Benhabib, Schmitt-Grohé and Uribe (2001) showed that, if the central bank follows a Taylor rule, then self-fulfilling expectations can induce the economy to fall into a permanent liquidity trap. Benigno and Fornaro (2015) proposed a theory of "stagnation traps" where the weakness of aggregate demand depresses investment in innovation, resulting in a low rate of technological progress. This pushes the nominal interest rate against the zero lower bound, which explains why aggregate demand remains so weak. In that context, subsidies to investment in innovation are welfare enhancing.

The optimal policy that I derive is similar to the "unconventional fiscal policy" of Correia, Farhi, Nicolini and Teles (2013). They showed that, in a new Keynesian model, it is possible to replicate a negative nominal interest rate by implementing a rising path of consumption taxes, to induce agents to front-load their demand for consumption, together with a falling path of labor income taxes, to offset the resulting distortion to labor supply. In the presence of capital, this policy must be supplemented with a temporary capital subsidy. However, they only considered a temporary liquidity trap. By contrast, under secular stagnation, the economy is permanently liquidity trapped. I therefore restrict my attention to stationary policies. Eggertsson (2010) also showed that some fiscal instruments can be relied upon to stimulate aggregate demand.

The paper is organized as follows. Section 2 presents the representative household model of secular stagnation. In Section 3, I discuss some important properties of the secular stagnation equilibrium. In Section 4, I investigate the effectiveness of conventional macroeconomic stabilization policies. The optimal policy is derived in Section 5. The implications of heterogeneity are briefly discussed in Section 6. The paper ends with a conclusion.

2 The Representative Household Model

2.1 Households

Time is continuous. There is a mass 1 of infinitely lived households. Each household consumes a quantity c_t of a single consumption good with price P_t. It supplies labor l_t paid at nominal wage W_t and receives a real lump-sum transfer ψ_t. The nominal wealth A_t of a household is composed of physical capital K_t, government bonds B_t, and money M_t:

$$A_t = P_t^K K_t + B_t + M_t, \tag{1}$$

where P_t^K denotes the nominal price of capital. Firms pay R_t to rent capital from households. Capital depreciates at rate δ. There are three components to the nominal return from a unit of physical capital: the rent R_t, the loss due to depreciation δP_t^K, and the capital gain $\dot{P}_t^K$. Bonds yield a nominal return i_t, while money yields a zero nominal return. The wealth of the representative household therefore evolves according to:

$$\begin{aligned} \dot{A}_t &= \left[R_t - \delta P_t^K + \dot{P}_t^K\right] K_t + i_t B_t + W_t l_t + P_t \psi_t - P_t c_t, \\ &= i_t A_t + \left[\frac{R_t}{P_t^K} - \delta + \frac{\dot{P}_t^K}{P_t^K} - i_t\right] P_t^K K_t - i_t M_t + W_t l_t + P_t \psi_t - P_t c_t. \end{aligned} \tag{2}$$

Let $\pi_t = \dot{P}_t / P_t$ denote the rate of inflation, $r_t = i_t - \pi_t$ the real interest rate, $w_t = W_t / P_t$ the real wage, $a_t = A_t / P_t$ the real household wealth, $m_t = M_t / P_t$ the real money holdings, and $p_t^K = P_t^K / P_t$ the real price of capital, measured in units of consumption good. Dividing the wealth accumulation equation by the price level P_t, while using the fact that $\dot{A}_t / P_t = \dot{a}_t + \pi_t a_t$ and $\dot{P}_t^K / P_t^K = \dot{p}_t^K / p_t^K + \pi_t$, yields:

$$\dot{a}_t = r_t a_t + \left[\frac{R_t}{P_t^K} - \delta + \frac{\dot{p}_t^K}{p_t^K} - r_t\right] p_t^K K_t - i_t m_t + w_t l_t + \psi_t - c_t. \tag{3}$$

In the absence of uncertainty, by arbitrage, the returns from holding capital must be equal to the returns from holding bonds. Thus, the second term on the right hand side of (3) must be equal to zero, which yields the following relationship between the user cost of capital and the real interest rate:

$$r_t = \frac{R_t}{P_t^K} - \delta + \frac{\dot{p}_t^K}{p_t^K}. \tag{4}$$

The wealth accumulation equation therefore simplifies to:

$$\dot{a}_t = r_t a_t - i_t m_t + w_t l_t + \psi_t - c_t. \tag{5}$$

The intertemporal budget constraint prevents households from running Ponzi schemes:

$$\lim_{t\to\infty} e^{-\int_0^t r_s ds} a_t \geq 0. \tag{6}$$

The representative household discounts the future at rate $\rho > 0$. At any point in time, it derives utility $u(c_t)$ from consuming c_t, with $u'(\cdot) > 0$, $u''(\cdot) < 0$, and $\lim_{c\to 0} u'(c) = \infty$. It also derives utility $h(m_t)$ from holding real money balances m_t, with $h'(\cdot) > 0$, $h''(\cdot) < 0$, $\lim_{m\to 0} h'(m) = \infty$, and $h'(m) = 0$ for all $m \geq \bar{m}$. At $\bar{m}$, the household is satiated with real money balances and does not derive any utility from holding more money for transaction purposes. The household incurs disutility $v(l_t)$ from supplying labor l_t, with $v'(\cdot) > 0$, $v''(\cdot) > 0$, $v'(0) = 0$, and $\lim_{l\to\bar{l}} v'(l) = \infty$ where $\bar{l}$ is the maximum feasible supply of labor, which can be infinite. The household also derives utility from holding wealth a_t. However, it knows that it will eventually need to cover the liabilities of the government, which are composed of the real aggregate supply of public debt b_t^s and of money m_t^s. The utility derived by a household from holding wealth is therefore equal to $\gamma(a_t - (b_t^s + m_t^s))$, with $\gamma'(\cdot) > 0$, $\gamma''(\cdot) \leq 0$, and $\lim_{K\to 0} \gamma'(K) < \infty$. Finally, at any time t, utility is additively separable between its four components. The intertemporal utility function is therefore given by:

$$\int_0^\infty e^{-\rho t} \left[u(c_t) - v(l_t) + h(m_t) + \gamma(a_t - (b_t^s + m_t^s)) \right] dt. \tag{7}$$

Including the government liabilities $b_t^s + m_t^s$ into the utility function ensures that the Ricardian equivalence holds and that the model has reasonable welfare properties. Indeed, if households did derive utility from holding wealth independently of the level of government liabilities, then the government would be able to artificially increase welfare by giving households a huge lump-sum subsidy that would eventually be offset by a huge lump-sum tax. This would increase both the level of public debt b_t^s and the wealth a_t of households and, hence, their welfare. At the zero lower bound, money and bonds are perfect substitutes and welfare could also be artificially increased by raising the aggregate money supply m_t^s. I therefore assume that the government cannot mechanically raise welfare by increasing the level of public liabilities. Note that, in the interesting special case where the marginal utility of wealth is constant, the behavior of households is identical whether or not government liabilities are included in the utility function.

The household's problem is to maximize intertemporal utility, (7), subject to the budget constraint, (5) and (6), with a_0 given. Importantly, the paths of public debt b_t^s and of aggregate money supply m_t^s from time 0 onwards are exogenous to the representative household's actions. By the maximum principle, the solution to the household's problem is characterized by:

$$\frac{\dot{c}_t}{c_t} = \left[r_t - \rho + \frac{\gamma'\left(a_t - \left(b_t^s + m_t^s\right)\right)}{u'\left(c_t\right)}\right] \frac{u'\left(c_t\right)}{-u''\left(c_t\right) c_t}, \tag{8}$$

$$v'\left(l_t\right) = w_t u'\left(c_t\right), \tag{9}$$

$$h'\left(m_t\right) = i_t u'\left(c_t\right), \tag{10}$$

together with the transversality condition:

$$\lim_{t \to \infty} e^{-\rho t} u'\left(c_t\right) a_t = 0. \tag{11}$$

When $i_t = 0$, there is no opportunity cost of holding money rather than bonds. In that case, the household's utility from real money balances must be satiated. Moreover, when $i_t = 0$, money and bonds are both zero interest yielding assets that can be used for saving, i.e. they are perfect substitutes. The household is therefore happy to rely on money for saving and, hence, its demand for real money balances can be anything greater or equal to $\bar{m}$. The zero lower bound on the nominal interest rate i_t therefore follows from the money demand equation (10).

2.2 Firms

Firms rent capital K_t^d from households and employ labor L_t to produce output y_t using a constant returns to scale neoclassical production function:

$$y_t = F\left(K_t^d, L_t\right). \tag{12}$$

They choose their demand for capital K_t^d and for labor L_t such as to maximize their profits:

$$P_t F\left(K_t^d, L_t\right) - R_t K_t^d - W_t L_t. \tag{13}$$

In equilibrium, each factor of production must be paid its marginal product:

$$\frac{R_t}{P_t} = F_K\left(K_t^d, L_t\right), \tag{14}$$

$$w_t = \frac{W_t}{P_t} = F_L\left(K_t^d, L_t\right). \tag{15}$$

By assumption, one unit of output can either be transformed into a consumption good or an investment good.[4] Hence, if investment is strictly positive, then the price of capital must be equal to that of consumption, i.e. $p_t^K = P_t^K / P_t = 1$. In that case, by (4) and

[4] However, investment goods cannot be transformed into consumption goods, i.e. investment cannot be negative.

(14), the real interest rate must be given by:

$$r_t = F_K\left(K_t^d, L_t\right) - \delta. \tag{16}$$

This expression applies in steady state where investment must be strictly positive to compensate the depreciation of capital.

2.3 Government

Let M_t^s and m_t^s denote the nominal and real money supply at time t, respectively. The growth rate of the nominal money supply at t is given by $\phi_t = \dot{M}_t^s/M_t^s$. We therefore have $M_t^s = M_0 e^{\int_0^t \phi_s ds}$, where the initial money supply M_0 is assumed to be exogenously given.[5] The supply of money by the government is therefore characterized by ϕ_t for all $t \in [0, +\infty)$.

At any time t, the government gets real revenue $\dot{M}_t^s/P_t$ from seigniorage, it incurs real expenditures g_t, and it makes a lump-sum transfer ψ_t to each household. Thus, the nominal supply B_t^s of government bonds evolves according to:

$$\dot{B}_t^s = i_t B_t^s + P_t \psi_t + P_t g_t - \dot{M}_t^s. \tag{17}$$

Thus, using the fact that $\dot{B}_t^s/P_t = \dot{b}_t^s + \pi b_t^s$, the real level of public debt $b_t^s = B_t^s/P_t$ evolves according to:

$$\begin{aligned} \dot{b}_t^s &= (i_t - \pi_t)\, b_t^s + \psi_t + g_t - \frac{\dot{M}_t^s}{M_t^s}\frac{M_t^s}{P_t}, \\ &= r_t b_t^s + \psi_t + g_t - \phi_t m_t^s. \end{aligned} \tag{18}$$

Note that $\phi_t m_t^s = \dot{M}_t^s/P_t = \dot{m}_t^s + \pi_t m_t^s$. Thus, expression (18) for the accumulation of public debt can also be written as:

$$\dot{b}_t^s + \dot{m}_t^s = r_t\left[b_t^s + m_t^s\right] + \psi_t + g_t - i_t m_t^s. \tag{19}$$

Integrating the debt accumulation equation (18) gives:

$$e^{-\int_0^t r_x dx} b_t^s = b_0 + \int_0^t e^{-\int_0^s r_x dx}\left[\psi_s + g_s - \phi_s m_s^s\right] ds, \tag{20}$$

[5]If the government can implement a discrete increase in the money supply at time 0, then it can effectively choose the initial price level P_0 such as to dilute the initial stock B_0 of government debt. Thus, taking M_0 as given in the context of monetary policy is like preventing the taxation of initial capital in the context of fiscal policy.

where $b_0 = B_0/P_0$ with B_0 exogenously given. Integrating instead equation (19) yields:

$$e^{-\int_0^t r_x dx} \left(b_t^s + m_t^s\right) = b_0 + m_0 + \int_0^t e^{-\int_0^s r_x dx} \left[\psi_s + g_s - i_s m_s^s\right] ds. \tag{21}$$

In this setup, we can consider two versions of the intertemporal government budget constraint. The first is given by the following no-Ponzi condition:

$$\lim_{t\to\infty} e^{-\int_0^t r_x dx} \left(b_t^s + m_t^s\right) \le 0. \tag{22}$$

This implies that money is a liability to the government, which must therefore asymptotically have enough wealth to be able to redeem the outstanding money supply. This is the most conservative version of the intertemporal budget constraint. It corresponds to a government that behaves responsibly and that always wants to be in full control of the money supply. Hence, seigniorage cannot be a permanent source of revenue. Indeed, as can be seen from (21), by issuing money rather than bonds, the government only economizes on the nominal interest rate. This no-Ponzi condition (22) is the most common form of the intertemporal government budget constraint found in the literature (see, for instance, Kocherlakota and Phelan 1999).

The lax version of the intertemporal government budget constraint is given by:

$$\lim_{t\to\infty} e^{-\int_0^t r_x dx} b_t^s \le 0. \tag{23}$$

Thus, asymptotically, the present value of debt cannot be positive. However, this does not put any restriction on the ability of the government to print money in order to pay down its debts. Indeed, equation (20) shows that, under the no-Ponzi condition (23), seigniorage is a source of public revenue. This implies that a government that issues debt in its own currency cannot be insolvent. Buiter (2014) forcefully argues that this is the intertemporal budget constraint that governments legally face as they do not have any obligation to redeem the outstanding money supply.

For most of my analysis, I will consider the conservative version of the government budget constraint, given by (22). Indeed, a government with a strong commitment to keeping inflation low wants to be able to redeem its outstanding money supply under all circumstances. Moreover, I want to characterize policies that can lift the economy out of depression without resulting in a huge accumulation of money or public debt that would be very dangerous should economic circumstances unexpectedly change in the future (as would be the case if households suddenly ceased to have a preference for wealth). I will nevertheless occasionally comment on the consequences of the lax version of the government budget constraint.

2.4 Market Clearing

In a frictionless equilibrium, five market clearing conditions must be satisfied. In the market for goods, production must be equal to sum of consumption, investment, and government expenditures:

$$y_t = c_t + \delta K_t + \dot{K}_t + g_t, \tag{24}$$

where investment is equal to $\delta K_t + \dot{K}_t$. In the labor market, firms' labor demand L_t must be equal to workers' labor supply l_t:

$$L_t = l_t. \tag{25}$$

The market for physical capital clears when the quantity K_t supplied by households through their savings is equal to the quantity K_t^d demanded by firms:

$$K_t = K_t^d. \tag{26}$$

In the bond market, households' demand for bonds B_t must be equal to the government supply B_t^s:

$$B_t = B_t^s. \tag{27}$$

Finally, in equilibrium, the market for money must also clear. The household's nominal demand for money is equal to $P_t m_t$, while the supply of money by the government is equal to $M_t^s = M_0 e^{\int_0^t \phi_s ds}$. Thus, the market for money is in equilibrium provided that:

$$m_t = e^{\int_0^t (\phi_s - \pi_s) ds} M_0 / P_0, \tag{28}$$

where I have used the fact that $P_t = P_0 e^{\int_0^t \pi_s ds}$.

2.5 Frictions

If prices, interest rates, and wages were unconstrained, then the five market clearing conditions would be satisfied. However, to obtain a stationary secular stagnation equilibrium, where the economy fails to produce at full capacity, I impose some restrictions on the adjustment of interest rates and of wages.

In an economy with fiat currency, a fundamental constraint on the equilibrium is that the nominal interest rate cannot be negative:

$$i_t \geq 0. \tag{29}$$

Indeed, no-one is willing to pay more than 100$ for a bond yielding 100$ in a year. People would rather choose to rely on money for saving. This follows from the money demand equation (10).

Prices are perfectly flexible. The central bank never allows inflation to rise above a ceiling π^*. Thus, in equilibrium, we must have:

$$\pi_t \leq \pi^*. \tag{30}$$

The combination of the zero lower bound on the nominal interest rate and of the upper bound on inflation results in a lower bound on the real interest rate:

$$r_t \geq -\pi^*. \tag{31}$$

Note that nothing in the model justifies the existence of an upper bound on inflation. However, it is a key ingredient of secular stagnation that does accurately describe the behavior of all major central banks around the world.

If wages were perfectly flexible, then they would be given by the marginal product of labor (15) with labor demand L_t equal to labor supply l_t:

$$w_t = F_L\left(K_t, l_t\right). \tag{32}$$

However, a binding zero lower bound on the nominal interest rate could result in a deflationary feedback loop, whereby a lack of demand induces a fall in prices, which raises the real interest rate, which further reduces demand. This is inconsistent with the existence of a stationary equilibrium. Hence, to put a break on the extent of deflation, I assume that wages are downward rigid.

Recall that the real wage w_t must always be equal to the marginal product of labor $F_L\left(K_t, L_t\right)$, as otherwise firms can increase profits by adjusting employment. For fixed values of the stock of physical capital K_t and of labor demand L_t, the nominal wage $W_t = P_t F_L\left(K_t, L_t\right)$ increases by the rate of inflation π_t. I impose the following lower bound on wage growth for unchanged values of K_t and L_t:

$$\left(1 + \pi_t dt\right) W_t \geq \left(1 + \pi^R dt\right) W_t - \left(\alpha dt\right)\left[W_t - P_t F_L\left(K_t, l_t\right)\right], \tag{33}$$

where π^R denotes the reference rate of inflation used in the wage bargaining process and $\alpha \geq 0$ denotes the speed of convergence of the wage to the marginal product of labor at full employment.

I assume that $W_t \geq P_t F_L\left(K_t, l_t\right)$ and the wage rigidity constraint (33) must hold with complementary slackness. Thus, if $W_t > P_t F_L\left(K_t, l_t\right)$, then it must be due to the binding wage rigidity constraint. Workers would like their nominal wage to increase at rate π^R, but are willing to accept a smaller increase to reduce the gap between their current wage W_t and the marginal product of labor at full employment $P_t F_L(K_t, l_t)$. Conversely, if the wage rigidity constraint is not binding, then the wage W_t must be equal to the marginal product of labor at full employment $P_t F_L(K_t, l_t)$.

The lower bound on wage growth is specified for fixed values of K_t and L_t, i.e. the left hand side of the inequality is $(1 + \pi_t dt) W_t = (1 + \pi_t dt) P_t F_L(K_t, L_t) = P_{t+dt} F_L(K_t, L_t)$ rather than $W_{t+dt} = P_{t+dt} F_L(K_{t+dt}, L_{t+dt})$. I am therefore assuming that, in the wage bargaining process, workers are willing to adjust their wage rate if it reflects a change in the stock of physical capital or in the amount of labor employed by firms. Thus, workers accept a wage cut if firms employ them for more hours, which reduces the marginal product of labor. By contrast, without this adjustment, the fall in the marginal product of labor following an increase in L_t would need to be compensated by higher inflation for the nominal wage $P_t F_L(K_t, L_t)$ to remain unchanged. This would greatly complicate the out-of-steady-state dynamics of inflation.[6] Note that, in steady state, both K_t and L_t are constant and these considerations are therefore irrelevant.

Two natural benchmarks are $\pi^R = 0$, which corresponds to a reluctance by workers to accept nominal wage cuts, and $\pi^R = \pi^*$, in which case workers would like their nominal wage to increase by the inflation ceiling, which might also be the central bank's inflation target.[7] Throughout my analysis, I shall consider that:

$$\pi^R \leq \pi^*. \tag{34}$$

Thus, workers never bargain for nominal wages that increase faster than the inflation ceiling. If $\alpha = 0$, then nominal wages for given values of K_t and L_t cannot increase by less than the reference rate of inflation π^R, independently of the amount l_t of labor supplied; while, as $\alpha \to \infty$, wages become perfectly flexible.

The wage rigidity constraint (33) can be written in real terms as:

$$\left[\pi_t - \left(\pi^R - \alpha\right)\right] w_t \geq \alpha F_L(K_t, l_t). \tag{35}$$

In equilibrium, we must have $\pi_t \in \left(\pi^R - \alpha, +\infty\right)$ as, otherwise, the constraint cannot be satisfied. The downward wage rigidity assumption is that (35) and $w_t \geq F_L(K_t, l_t)$ must hold with complementary slackness. Thus, if the wage rigidity constraint is slack, then $w_t = F_L(K_t, l_t)$, which, by (35), implies $\pi_t \in \left(\pi^R, +\infty\right)$. Hence, if $\pi_t \in \left(\pi^R - \alpha, \pi^R\right]$, then the wage rigidity constraint must hold with equality. If $w_t > F_L(K_t, l_t)$, then the

[6] If α is close to zero and π^R close to π^*, then inflation might have to rise above the ceiling π^* to compensate for a fall in the marginal product of capital or labor. This can prevent the existence of an equilibrium.

[7] Eggertsson and Mehrotra (2014) impose a similar wage rigidity assumption, but assume that $\pi^R = 0$.

wage rigidity constraint must be binding, which implies that $\pi_t \in \left(\pi^R - \alpha, \pi^R\right)$. Hence, if $\pi_t \in \left[\pi^R, +\infty\right)$, then $w_t = F_L\left(K_t, l_t\right)$. The real wage must therefore satisfy:

$$w_t = \begin{cases} F_L\left(K_t, l_t\right) & \text{if } \pi_t \in \left[\pi^R, +\infty\right) \\ \frac{\alpha}{\pi_t - (\pi^R - \alpha)} F_L\left(K_t, l_t\right) & \text{if } \pi_t \in \left(\pi^R - \alpha, \pi^R\right) \end{cases} \tag{36}$$

Recall that we always have $w_t = F_L\left(K_t, L_t\right)$. Hence, if inflation is above the reference rate π^R of inflation, then the downward wage rigidity is not binding and the labor market clears, i.e. $L_t = l_t$. If, however, inflation is below the reference rate π^R, then the downward wage rigidity maintains the real wage above the marginal product of labor at full employment, which generates a labor demand L_t that falls short of the labor supply l_t. For a given rate π_t of inflation smaller than π^R, a rise in wage flexibility as measured by α reduces the discrepancy between labor demand and labor supply.

2.6 Equilibrium

An equilibrium is defined as follows.

Definition 1 *An equilibrium consists of paths of prices P_0 and $\left(r_t, i_t, \pi_t, w_t, R_t, p_t^K\right)_{t=0}^{\infty}$, of quantities $\left(c_t, l_t, K_t, b_t, y_t, L_t, K_t^d, m_t\right)_{t=0}^{\infty}$, and of a government policy $\left(\psi_t, g_t, \phi_t, b_t^s\right)_{t=0}^{\infty}$ such that:*

- $r_t = i_t - \pi_t$ *with* $i_t \geq 0$ *and* $\pi_t \leq \pi^*$;
- $\left(c_t, l_t, K_t, b_t, m_t\right)$ *solves the consumer's problem given* $\left(r_t, i_t, w_t, R_t, p_t^K, \psi_t\right)_{t=0}^{\infty}$, K_0, $b_0 = B_0/P_0$, *and* $m_0 = M_0/P_0$;
- $\left(y_t, L_t, K_t^d\right)_{t=0}^{\infty}$ *solves the producer's problem given* P_0 *and* $\left(\pi_t, w_t, R_t\right)_{t=0}^{\infty}$;
- $\left(\psi_t, g_t, \phi_t, b_t^s\right)_{t=0}^{\infty}$ *satisfies the intertemporal government budget constraint given* $\left(r_t, i_t, \pi_t\right)_{t=0}^{\infty}$, $b_0 = B_0/P_0$, *and* $m_0 = M_0/P_0$;
- *Equilibrium prices* P_0 *and* $\left(r_t, i_t, \pi_t, w_t, R_t, p_t^K\right)_{t=0}^{\infty}$ *are such that markets clear:*
  - *The market for goods clears:* $y_t = c_t + \delta K_t + \dot{K}_t + g_t$;
  - *The market for physical capital clears:* $K_t = K_t^d$;
  - *The bond market clears:* $b_t = b_t^s$;
  - *The market for money clears:* $m_t = e^{\int_0^t (\phi_s - \pi_s) ds} M_0/P_0$;
  - *The real wage is given by (36).*[8]

[8] Clearly, the real wage being equal to both (15) and (36), the labor market does not necessarily clear in equilibrium.

I first characterize the steady state equilibria of this economy. I then discuss their stability. Throughout this section, I set the government expenditures to zero, i.e. $g_t = 0$, and impose the conservative version of the intertemporal government budget constraint, given by (22). I take the growth rate ϕ_t of the money supply as given and subsequently characterize the conditions that it needs to satisfy in equilibrium. I assume that the lump-sum transfer ψ_t is constant over time and that it adjusts such as to have a binding intertemporal government budget constraint given $(i_t, r_t, \pi_t)_{t=0}^{\infty}$, $b_0 = B_0/P_0$, and $m_0 = M_0/P_0$.[9] The equilibrium level of government debt b_t is determined by the resulting policy, according to (20).

2.6.1 Steady State Equilibria

In a steady state equilibrium, consumption c_t, labor supply l_t, labor demand L_t, physical capital K_t, the real wage w_t, and the real interest rate r_t must all be constant over time. Without loss of generality, we can also consider that both inflation π_t and the nominal interest rate i_t are constant over time.[10] I subsequently drop the time subscripts from those variables to refer to their steady state values.

For the capital stock to be constant, investment needs to be strictly positive to offset depreciation. In steady state, the real price of capital p_t^K must therefore be equal to 1. Hence, the real interest rate is just equal to the marginal product of capital net of depreciation, cf. (16). Note that, by (1), when $p_t^K = 1$, wealth net of government liabilities $a_t - (b_t + m_t)$ must be equal to the stock of physical capital K.

The simplest way to characterize the steady state equilibria of the economy is to look at it through the lens of the Aggregate Demand - Aggregate Supply paradigm. Fundamentally, aggregate demand is given by $y = c + \delta K$, while aggregate supply is given by $y = F(K, L)$. I therefore need compute these levels of demand and supply for each rate π of inflation. For this, I will rely on the characterization of (c, l, L, K, w) as a function of the real interest rate r, which is jointly obtained from the optimality conditions for consumption (8) with $\dot{c}_t = 0$ and $a_t - (b_t + m_t) = K$ and for labor supply (9), from the demand for labor (15) and for capital (16), and from the expression for the real wage (36).

Clearly, if inflation is high enough, then the zero lower bound on the nominal interest rate is not binding and the real interest rate can be sufficiently small to induce the economy to produce at full capacity. Let $\bar{\pi}$ denote the smallest rate of inflation such

[9] Recall that m_t^s is equal to $e^{\int_0^t (\phi_s - \pi_s) ds} M_0/P_0$, which is why ϕ_t and π_t do enter the intertemporal government budget constraint.

[10] If $i_t = 0$, then $r_t = -\pi_t$. If $\pi_t < \pi^R$, then inflation determines the gap between w_t and $F_L(K_t, l_t)$, through (36). Thus, in these two cases, steady state inflation must be constant. And, if $i_t > 0$ and $\pi_t \geq \pi^R$, then money is super-neutral. Hence, there is no loss of generality in considering that inflation is constant over time.

that the zero lower bound is not binding. I assume that $\bar{\pi} > \pi^*$, which implies that the inflation ceiling π^* is not high enough to allow the economy to produce at full capacity. Recall that, by (34), the reference rate of inflation π^R is assumed to be lower than the inflation ceiling π^*. We therefore have:

$$\bar{\pi} > \pi^* \geq \pi^R. \tag{37}$$

By (15) and (36), the labor market clears, i.e. $L = l$, when $\pi \in [\pi^R, +\infty)$, but fails to clear, i.e. $L < l$, when $\pi \in (\pi^R - \alpha, \pi^R)$. We can therefore distinguish three intervals of interest. When $\pi \in [\bar{\pi}, +\infty)$, the frictions are irrelevant. When $\pi \in [\pi^R, \bar{\pi})$, the zero lower bound is binding but the labor market clears. Finally, when $\pi \in (\pi^R - \alpha, \pi^R)$, the zero lower bound is binding and the labor market does not clear.[11]

Thus, when $\pi \in [\bar{\pi}, +\infty)$, the economy must be in a frictionless equilibrium.

Lemma 1 *A steady state frictionless equilibrium always exists and, if ρ is sufficiently close to zero and labor supply is sufficiently inelastic, then it must be unique.*

Note that, even if ρ is high and labor supply is highly elastic, it is difficult to find a counter-example with multiple equilibria. I therefore consider throughout my analysis that the frictionless equilibrium is unique.

Let r^n denote the natural real interest rate, i.e. the real interest rate of the frictionless equilibrium. Recall that (c, l, L, K, w) can be characterized as a function of the real interest rate. The natural rate r^n is therefore determined such as to equate aggregate demand $y = c + \delta K$ to aggregate supply $y = F(K, L)$. The corresponding nominal interest rate is given by $i = r^n + \pi$. By definition, $\bar{\pi}$ is the smallest rate of inflation at which frictions are irrelevant. Hence, when inflation is equal to $\bar{\pi}$, we must have $i = 0$. This implies that $\bar{\pi} = -r^n$.[12]

When $\bar{\pi} > \pi^*$, the frictionless equilibrium is inconsistent with the inflation ceiling π^*. The condition that $\bar{\pi} > \pi^*$ is fundamentally an assumption on the natural real interest rate, which must satisfy $r^n = -\bar{\pi} < -\pi^*$. In particular, if $\pi^* \geq 0$, the natural rate must be negative. The optimality condition for consumption (8) implies that, in steady state, $r = \rho - \gamma'(K)/u'(c)$. Hence, if ρ is sufficiently small, the real interest rate is indeed likely to be negative. This shows that the preference for wealth decreases the natural real interest rate to such an extent that it can easily make the frictionless equilibrium inconsistent with the inflation ceiling π^*.

[11] Recall that, by the wage rigidity constraint (35), $\pi \leq \pi^R - \alpha$ is inconsistent with the existence of an equilibrium.

[12] It can easily be shown that $\bar{\pi} < \delta$. In equilibrium, the marginal product of capital $F_K(K, L)$ must be strictly positive. Hence, by (16), we must have $r^n + \delta = F_K(K, L) > 0$, which implies that $\bar{\pi} = -r^n < \delta$.

When $\pi \in \left[\pi^R, \bar{\pi}\right)$, the labor market clears but the real interest is excessively high, as $r = i - \pi > 0 - \bar{\pi} = -r^n$. Under mild conditions, this is inconsistent with the existence of a steady state equilibrium.

Lemma 2 *Aggregate demand $y = c + \delta K$ is strictly smaller than aggregate supply $y = F(K, L)$ for all $\pi \in \left[\pi^R, \bar{\pi}\right)$ provided that ρ is sufficiently close to zero and either that labor supply is sufficiently inelastic or that $c > \rho K$ for all $r \in [r^n, \rho]$.*[13]

Note that, if consumption c is greater than investment δK and if $\delta > \rho$, then $c > \rho K$.

We can now prove the existence and uniqueness of a steady state equilibrium for $\pi \in \left(\pi^R - \alpha, \pi^R\right)$ provided that wages are sufficiently sticky, i.e. provided that α is sufficiently small. I consider that the growth rate of the money supply is sufficiently strong to have a binding zero lower bound in steady state equilibrium.[14] Indeed, aggregate demand would be even more depressed with a strictly positive nominal interest rate.

Lemma 3 *If aggregate demand is strictly smaller than aggregate supply when $\pi = \pi^R$ (as implied by Lemma 2) and if $\alpha < \pi^R + \rho$, then there exists at least one steady state equilibrium such that $\pi \in \left(\pi^R - \alpha, \pi^R\right)$. If α is sufficiently close to zero, this must be the unique steady state equilibrium with $i = 0$ and $\pi < \bar{\pi}$.*

I henceforth consider that the secular stagnation equilibrium is unique. When $\pi \in \left(\pi^R - \alpha, \pi^R\right)$, the wage rigidity constraint is binding, which implies that firms' labor demand L is smaller than households' labor supply l. This clearly shows that, in the secular stagnation equilibrium, the economy fails to produce at full capacity.

Figure 1 displays the aggregate demand and aggregate supply curves, assuming that $i = 0$ and $r = -\pi$ for all $\pi \in \left(\pi^R - \alpha, \bar{\pi}\right)$. When $\pi \in [\bar{\pi}, +\infty)$, the zero lower bound is not binding. The economy therefore is in the frictionless equilibrium where the real interest rate adjusts such as to equate aggregate demand to aggregate supply. Hence, the two curves overlap at the natural level of output for all $\pi \in [\bar{\pi}, +\infty)$.

By Lemma 2, we know that aggregate supply is greater than aggregate demand whenever $\pi \in \left[\pi^R, \bar{\pi}\right)$. Note that, as the real interest rate increases from $-\bar{\pi}$ to $-\pi^R$, aggregate supply can either rise or fall. On the one hand, an increase in the real rate reduces K/L, by (16), which contracts aggregate supply $y = F(K, L) = LF(K/L, 1)$. On the other

[13] The consumption Euler equation (8) implies that in steady state $r \leq \rho$, which is why we can ignore the possibility that $r > \rho$.

[14] A sufficient condition is $\phi \in \left[\pi^R, \bar{\pi}\right)$. Indeed, to have a steady state equilibrium with $i > 0$, inflation must be equal to the money growth rate, by (10) and (28). Hence, if $\pi = \phi \in \left[\pi^R, \bar{\pi}\right)$ and $i > 0$, then $r = i - \phi > -\phi > -\bar{\pi} = r^n$. But, by Lemma 2, there is no steady state equilibrium with $r > r^n$ and $\pi \in \left[\pi^R, \bar{\pi}\right)$. Hence, when $\phi \in \left[\pi^R, \bar{\pi}\right)$, there is no steady state equilibrium with $i > 0$.

hand, a higher real rate reduces the demand for consumption, by (8), which can raise the supply of labor, by (9), which expands aggregate supply, since $L = l$ when $\pi \geq \pi^R$.[15]

Finally, assuming that wages are sufficiently sticky, we know by Lemma 3 that an equilibrium always exists for some $\pi \in \left(\pi^R - \alpha, \pi^R\right)$. Indeed, as inflation falls from π^R to $\pi^R - \alpha$, labor demand L shrinks from l to 0, by (36). The capital stock K also falls in order to maintain a capital-labor ratio K/L consistent with the real interest rate $r = -\pi$, by (16). This generates a much stronger contraction in aggregate supply than in aggregate demand, as the fall in K and L does not affect the consumption component of aggregate demand.

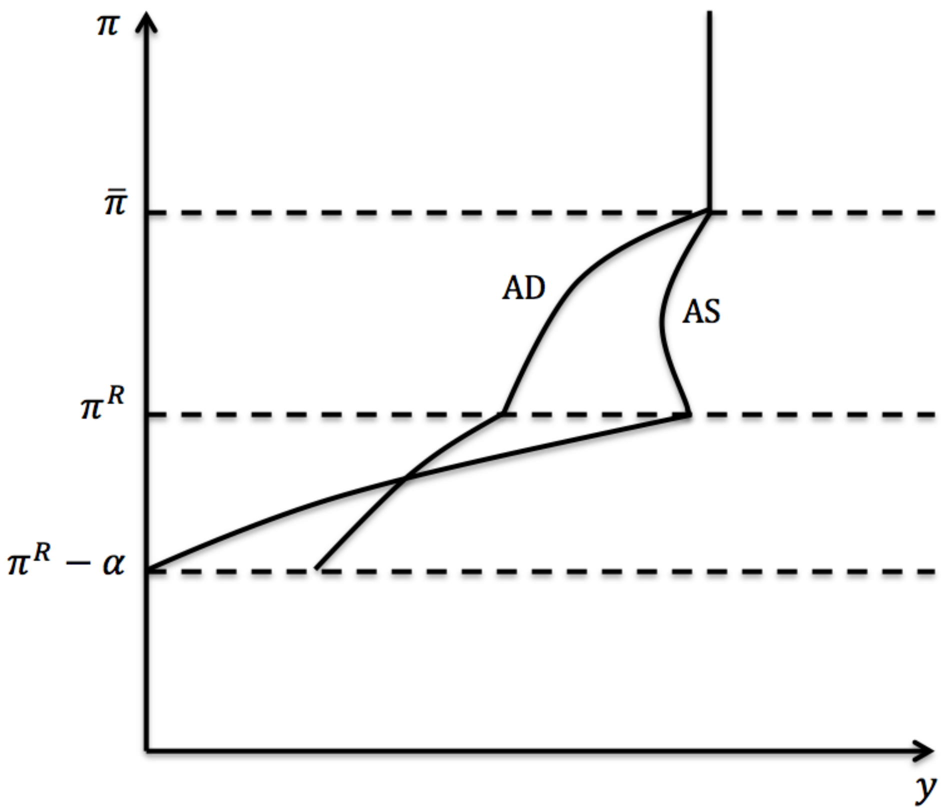

Figure 1: Aggregate Demand (AD) and Aggregate Supply (AS) curves

If the government does not allow inflation to rise above its ceiling $\pi^* \in \left[\pi^R, \bar{\pi}\right)$, then the only feasible steady state equilibrium of the economy is the secular stagnation equilibrium with $\pi \in \left(\pi^R - \alpha, \pi^R\right)$. The economy is therefore confronted with persistent under-employment, as $L < l$, and with an inflation rate that is persistently below the ceiling.

[15] In theory, aggregate demand can also be decreasing in inflation for some $\pi \in \left[\pi^R, \bar{\pi}\right)$. This is due to the investment component of aggregate demand, which is equal to $\delta\left(K/L\right)L$. However, this effect is unlikely to dominate as the consumption component typically is strongly increasing in inflation.

Let us now characterize the monetary policy that is consistent with the different equilibria of the economy. In a frictionless equilibrium with $\pi \in (\bar{\pi}, +\infty)$, the nominal interest rate is constant and strictly positive, as $i = r^n + \pi > r^n + \bar{\pi} = 0$. The money demand equation (10) therefore implies that real money balances m_t must also be constant over time. But, by the money market clearing condition (28), this requires the growth rate ϕ_t of the money supply to be equal to the inflation rate π. We must therefore have $\phi = \pi \in (\bar{\pi}, +\infty)$ The initial price level P_0 is then trivially determined by:

$$P_0 = \frac{M_0}{m}, \tag{38}$$

where m is given by the money demand equation (10) with $i = r^n + \pi$, while M_0 is exogenously given.[16]

In the secular stagnation equilibrium, the zero lower bound is binding. Thus, the money demand equation (10) implies that $m_t \geq \bar{m}$. The initial price level must therefore satisfy $M_0/P_0 \geq \bar{m}$ or, equivalently, $P_0 \leq M_0/\bar{m}$. Indeed, when $i = 0$, money and bonds are perfect substitutes. Hence, while the transaction demand for money never rises above $\bar{m}$, the extra money supplied $m_t - \bar{m} \geq 0$ can be used for savings. Assuming that, initially, households exclusively rely on bonds for savings, the price level must be given by:

$$P_0 = \frac{M_0}{\bar{m}}. \tag{39}$$

In that case, to have an equilibrium with $i = 0$, the real supply of money $e^{\int_0^t (\phi_s - \pi) ds} M_0/P_0 = e^{\int_0^t (\phi_s - \pi) ds} \bar{m}$ must always be greater or equal to $\bar{m}$. Hence, such an equilibrium with a given rate $\pi \in \left(\pi^R - \alpha, \pi^R\right)$ of inflation is consistent with any path of the growth rate of the money supply that satisfies $\int_0^t (\phi_s - \pi)\, ds \geq 0$ for all t. Thus, if the central bank adopts a constant growth rate ϕ of the money supply, then a liquidity trap equilibrium with inflation $\pi \in \left(\pi^R - \alpha, \pi^R\right)$ requires $\phi \geq \pi$.

The last remaining possibility is to have a frictionless equilibrium with inflation exactly equal to $\bar{\pi}$. In that case, the nominal interest rate is equal to zero. Hence, in theory, a rise in the money supply is not necessarily inflationary. However, if a strictly positive mass of households decides to spend the extra money supplied, rather than save it, then inflation and the nominal interest rate both rise. Thus, the liquidity trap equilibrium with inflation equal to $\bar{\pi}$ is not stable. This is fundamentally due to the fact that, although

[16] It is well known (cf. Obstfeld and Rogoff 1983) that several paths of the price level are consistent with a given path of the money supply. I choose to focus on the path of the price level that results in a constant rate of inflation. This equilibrium selection is consistent with the quantity theory of money as a once-and-for-all increase in the money supply results in a once-and-for-all increase in the price level of the same magnitude (cf. Kocherlakota and Phelan 1999). But, by (10), (28), and $i_t = r^n + \pi_t$, any path of π_t and ϕ_t that satisfy $h'\left(e^{\int_0^t (\phi_s - \pi_s) ds} M_0/P_0\right) = [r^n + \pi_t]\, u'(c)$ with $\pi_t \geq \bar{\pi}$ for all t is consistent with the steady state frictionless equilibrium.

the nominal interest rate is equal to zero, the zero lower bound is not binding as it is not preventing the nominal rate from being negative. We can therefore consider that, in a frictionless equilibrium with $i = 0$ and $\pi = \bar{\pi}$, the real quantity of money must be exactly equal to $\bar{m}$ and the growth rate of the money supply must be exactly equal to $\bar{\pi}$. In that case, the initial price level P_0 is equal to $M_0/\bar{m}$.

Finally, we need to check that the representative household's intertemporal budget constraint (6) and transversality condition (11) are both satisfied. Recall that the lump-sum transfer adjusts such as to balance the intertemporal government budget constraint. It follows that the government's no-Ponzi condition (22) holds with equality. But, in steady state, the real interest rate is constant and the wealth of the representative household is given by $a_t = K + b_t + m_t$. It immediately follows that:

$$\begin{aligned} \lim_{t\to\infty} e^{-\int_0^t r_s ds} a_t &= \lim_{t\to\infty} e^{-rt} K + \lim_{t\to\infty} e^{-rt} \left(b_t + m_t\right), \\ &= \lim_{t\to\infty} e^{-rt} K \geq 0. \end{aligned} \tag{40}$$

The household's intertemporal budget constraint is therefore binding when $r > 0$, but not when $r \leq 0$.

The optimality condition for consumption (8) implies that, in steady state, $\rho \geq r$. It follows that:

$$\begin{aligned} \lim_{t\to\infty} e^{-\rho t} u'(c_t)\, a_t &= \lim_{t\to\infty} e^{-\rho t} u'(c) \left(K + b_t + m_t\right), \\ &= \lim_{t\to\infty} e^{-\rho t} u'(c)\, K + \lim_{t\to\infty} e^{-(\rho - r)t} u'(c)\, e^{-rt} \left(b_t + m_t\right), \\ &= 0 + \left(\lim_{t\to\infty} e^{-(\rho - r)t} u'(c)\right) \left(\lim_{t\to\infty} e^{-rt} \left(b_t + m_t\right)\right), \\ &= 0. \end{aligned} \tag{41}$$

Thus, the household's transversality condition is always satisfied, even when its budget constraint is not binding. This confirms the optimality of the behavior of the representative household.

2.6.2 Stability of Equilibrium

We have shown that there are essentially two steady state equilibria: a frictionless steady state that requires inflation to be greater or equal to $\bar{\pi}$ and a secular stagnation steady state with inflation smaller than π^R. I now investigate the stability properties of these equilibria.

In a world with no lower bound on the real interest rate and no downward wage rigidity, any steady state equilibrium must be frictionless. If there is a unique frictionless steady state and if labor supply is inelastic, then a phase diagram reveals that the frictionless equilibrium must be globally saddle-path stable. By continuity, this must also be true if labor supply is sufficiently inelastic.[17]

Let us now turn to the stability of the secular stagnation steady state, which is the only feasible steady state equilibrium under the inflation ceiling π^*.[18] If the zero lower bound and the downward wage rigidity are both binding and if the real price p_t^K of capital is equal to 1, then the secular stagnation steady state is locally unstable provided that α is close to zero, i.e. that wages are very rigid, and that in steady state $-K\gamma''(K)/\gamma'(K) < -cu''(c)/u'(c)$.[19] This can easily be seen in the special case where α is equal to 0, which implies that inflation must be equal to π^R, and where the marginal utility of wealth is constant, i.e. $\gamma''(K) = 0$. Let us assume for a contradiction that the economy converges to its secular stagnation steady state. With constant inflation and constant marginal utility of wealth, the consumption Euler equation (8) implies that consumption immediately jumps to its steady state value c. Similarly, the demand for capital (16) implies that the capital-labor ratio K_t/L_t also immediately jumps to its steady state value K/L. The capital accumulation equation could then be written as:

$$\begin{aligned} \dot{K}_t &= F(K_t, L_t) - \delta K_t - c_t, \\ &= K_t \left[F\left(1, \frac{1}{K/L}\right) - \delta \right] - c. \end{aligned} \tag{42}$$

Clearly, if K_t is above its steady state value K, then K_t must diverge off to infinity; while if K_t is below K, then it must diverge towards zero.

However, physical capital is the only state variable of the economy. Also, if there is an upper bound to labor supply, then physical capital must belong to a closed interval. Hence, if an equilibrium exists, then physical capital must be converging towards a steady state.[20] It follows that in the neighborhood of the secular stagnation steady state, we must either have a non-binding wage rigidity constraint or $p_t^K < 1$.[21]

Let us first consider the scenario where the economy is in the frictionless equilibrium with inflation $\bar{\pi}$ and where the central bank unexpectedly lowers the inflation ceiling to π^*. The real interest rate immediately increases to $-\pi^*$, which is higher than the marginal

[17] It can also be shown, by linearizing (8) with $r_t = F_K(K_t, l_t) - \delta$ and $a_t - (b_t + m_t) = K_t$ and (24) with $y_t = F(K_t, l_t)$ around the steady state (c, K) while using (9) with $w_t = F_L(K_t, l_t)$ to express l_t as a function of c_t and K_t that, if a steady state satisfies $r^n = F_K(K, l) - \delta \leq 0$ and $c > \rho K$, then it must be locally saddle-path stable.

[18] I am now implicitly assuming that the growth rate of the money supply is sufficiently high to avoid having a steady state with a positive nominal interest rate and $\pi < \bar{\pi}$. By footnote 14, a sufficient condition for this is $\phi \in [\pi^R, \bar{\pi})$.

[19] This can be shown by linearizing the system around the secular stagnation steady state.

[20] As in the standard Ramsey model, $K_t = 0$ is not a stable steady state equilibrium.

[21] Note that, as the zero lower bound is binding in the secular stagnation steady state, we cannot have a strictly positive nominal rate while the economy is asymptotically converging to that steady state. Hence, we cannot relax the assumption that $i_t = 0$, while considering that the wage rigidity constraint is binding and that $p_t^K = 1$.

product of capital net of depreciation. But, by arbitrage, the returns from holding bonds must always be equal to the returns from holding physical capital, cf. (4) and (14). The price p_t^K of capital must therefore fall below 1 such as to satisfy the following relationship between the real interest rate $-\pi_t$ and the marginal product of capital:

$$-\pi_t = \frac{F_K\left(K_t, L_t\right)}{p_t^K} - \delta + \frac{\dot{p}_t^K}{p_t^K}. \tag{43}$$

Aggregate investment drops to zero. The capital stock therefore falls are rate δ, since $\dot{K}_t = -\delta K_t$. Both the fall in the capital stock and the higher real interest rate reduce consumption. This results in a sharp contraction in aggregate demand and, therefore, in the demand for labor, which in the absence of investment is implicitly determined by $F\left(K_t, L_t\right) = c_t$. The fall in labor demand below labor supply induces inflation to drop below π^R.

The capital stock falls until it reaches the secular stagnation steady state. At this point, the price of capital becomes equal to 1, and investment becomes positive. This induces a discrete rise in labor demand, which raises the marginal product of capital and equates it to the steady state real interest rate.

If the initial stock of capital is below the secular stagnation steady state, then it must be rising. This requires positive investment and, hence, a price of capital p_t^K equal to 1. To avoid diverging away from the steady state, the downward wage rigidity must not be binding. Hence, the economy must be operating at full capacity, with $L = l$, while converging towards secular stagnation. The dynamics of the frictionless economy are at work, except that the economy is not on a path converging to the frictionless steady state. Once the capital stock reaches its secular stagnation value, the economy suddenly stalls. Indeed, investors know that any further increase in the capital stock will eventually result in the price of capital dropping below 1 and, hence, in a capital loss.

Once in secular stagnation, investment drops as $\dot{K}_t$, which was strictly positive, suddenly becomes equal to zero. This generates a fall in labor demand, which becomes smaller than labor supply. This raises the capital-labor ratio. The real interest rate therefore falls, such that in steady state $r = -\pi \in \left(-\pi^R, -\left(\pi^R - \alpha\right)\right)$. Importantly, this implies that the real interest rate just before falling into secular stagnation was higher than $-\pi^R$ and, hence, higher than the lower bound $-\pi^*$.

Interestingly, both the convergence from above or from below imply that the secular stagnation steady state is reached in finite time.

3 Properties of the Secular Stagnation Equilibrium

If the central bank does not allow inflation to rise above π^*, with $\pi^* < \bar{\pi} = -r^n$, then the economy must eventually fall into the secular stagnation steady state. Before investigating its properties, I begin by deriving a useful comparative statics result. We expect an exogenous rise in the real interest rate to reduce both consumption and output.[22] The following lemma provides a necessary and sufficient condition for this.

Lemma 4 *In the secular stagnation steady state, an exogenous rise in the real interest rate reduces both consumption and output if and only if* $-K\gamma''(K)/\gamma'(K) < -cu''(c)/u'(c)$.

Throughout my analysis, I therefore assume that, in the secular stagnation steady state, $-K\gamma''(K)/\gamma'(K) < -cu''(c)/u'(c)$. This condition must be satisfied for all the propositions of this section to hold. It does indeed seem natural to consider that the utility of wealth is less concave than the utility of consumption.

Under secular stagnation, the economy is permanently depressed with some unemployed labor resources. However, the fact that the labor demand of firms is persistently below the labor supply of workers is not fundamentally due to the wage rigidity. Indeed, when wages are more flexible, the economy is more depressed. This is known as the paradox of flexibility, which is formalized in the following proposition.

Proposition 1 *In the secular stagnation equilibrium, if α is sufficiently close to zero, then a rise in α, i.e. a rise in wage flexibility, reduces inflation, which lowers the output level.*

The intuition for the result is straightforward. If wages are more flexible, then deflation is stronger. This raises the real interest rate, which reduces the demand for consumption and for investment, resulting in a lower output level. Note that the sufficient condition of the proposition, that α is sufficiently close to zero, is far from necessary.[23]

The downward wage rigidity breaks the deflationary spiral and is therefore necessary to have a steady state equilibrium, as shown by Lemma 3. However, it is clearly not

[22] The exogenous rise in the steady state real interest rate can be due to a reduction in the growth rate of the money supply that raises the nominal rate. If α is sufficiently close to zero, i.e. if the interval $(\pi^R - \alpha, \pi^R)$ is sufficiently small, then the rise in the nominal interest rate i must also unambiguously raise the real interest rate $i - \pi$, since $\pi \in (\pi^R - \alpha, \pi^R)$.

[23] To overturn the result with an α significantly greater than zero, the elasticity of intertemporal substitution of consumption or the elasticity of labor of labor supply must be so large that a fall in inflation reduces consumption and, hence, labor demand or raises labor supply so much that inflation decreases by an even larger amount. Thus, in equilibrium, a rise in α must *raise* inflation such as to increase labor demand or reduce labor supply. This generates an upward pressure on inflation that is partially offset by the downward pressure that is mechanically triggered by the rise in α. This mechanism seems extremely implausible. A similar reasoning implies that Proposition 2, 3, and, 4, below, also are much more robust than the sufficient condition suggests.

the friction at the origin of the economic depression. Secular stagnation is instead due to a lack of demand resulting from the lower bound on the real interest rate. Indeed, the preference for wealth induces households to have such a high propensity to save that, in the absence of frictions, a very low real interest rate is necessary to raise aggregate demand to the level of aggregate supply. In a nutshell, the gap between labor supply and labor demand is not primarily due to the wage being too high, it is instead due to the real interest rate being too high.

The inflation ceiling π^* induces the economy to fall into secular stagnation, where the real interest rate $r = -\pi \in \left(-\pi^R, -\left(\pi^R - \alpha\right)\right)$ is essentially determined by the downward wage rigidity. At this interest rate, which is actually strictly higher than the lower bound $-\pi^*$, aggregate demand is very weak. Hence, the labor demand L of firms falls below the labor supply l of workers to equate aggregate supply to the weak level of aggregate demand. Output is essentially demand determined.

To understand why a lack of demand can reduce the output level of the economy, it is important to realize that a rise in the propensity to save does not necessarily translate into higher investment. In fact, under secular stagnation, a rise in the intensity of the preference for wealth reduces investment. This is known as the paradox of thrift, which is stated in the following proposition.

Proposition 2 *In the secular stagnation equilibrium, if α is sufficiently close to zero, then a rise in the marginal utility of wealth reduces both consumption and investment.*

Indeed, a rise in the propensity to save reduces consumption, which contracts the level of aggregate demand, which in turn reduces the labor demand of firms. In the limit as α tends to zero, the real interest rate remains equal to $-\pi^R$ and the capital-labor ratio must remain constant. This requires a fall in the capital stock and, hence, in steady state investment. If α is positive, the reduction in aggregate demand decreases inflation, which amplifies the contraction in both consumption and investment. Thus, a rise in the marginal utility of wealth lowers the equilibrium level of wealth!

By contrast, in the frictionless equilibrium, a rise in the supply of savings, due to an increase in the marginal utility of wealth, reduces the equilibrium real interest rate. This raises the stock of capital. It also raises consumption provided that, initially, the capital stock is not too high.[24] In the absence of frictions, we therefore have the usual result that a rise in the propensity to save raises investment and the capital stock. The fundamental

[24] By totally differentiating the equations that characterize the frictionless steady state equilibrium, it can be shown that a sufficient condition to have the capital stock K increasing in the marginal utility of wealth is that the natural real interest rate is negative (or positive but sufficiently close to zero) and either that labor supply is sufficiently inelastic or that $c > \rho K$. In that case, consumption c is also increasing in the marginal utility of wealth provided that the natural real interest rate is not too negative, i.e. provided that the capital stock is not too high.

problem under secular stagnation is that the real interest rate is determined by inflation, and hence by the downward wage rigidity, rather than by the supply and demand for loans.

When the economy suffers from a lack of demand, expansionary supply shocks can be contractionary. This is the paradox of toil, which is formalized in the following proposition.

Proposition 3 *In the secular stagnation equilibrium, if α is sufficiently close to zero:*

- *A fall in the disutility of labor reduces consumption, investment and, hence, output;*
- *Under a Cobb-Douglas aggregate production function, a rise in total factor productivity reduces consumption, investment and, hence, output.*

A rise in the supply of labor l, decreases the marginal product of labor at full capacity $F_L(K, l)$. By the binding wage rigidity constraint (33), this reduces the growth rate of nominal wages and, hence, the inflation rate. The corresponding rise in the real interest rate $r = -\pi$ generates a contraction in the demand for consumption and for investment.

The effect of a rise in total factor productivity is slightly more complex. On the one hand, it mechanically raises the marginal product of capital, which induces firms to raise their capital-labor ratio K/L. On the other hand, thanks to higher productivity and to more capital per worker, the amount of labor L necessary to meet the demand for consumption falls. It turns out that, with a Cobb-Douglas aggregate production function, these two effects exactly cancel out and, hence, total factor productivity has no direct impact on investment δK. However, a higher total factor productivity raises the marginal product of labor $F_L(K, L)$, which increases the labor supply l of workers.[25] As before, this reduces inflation, which generates a contraction in the demand for consumption and for investment.

In a nutshell, an expansionary supply shock cannot raise output if it fails to generate a corresponding increase in aggregate demand. The problem is that, under secular stagnation, the real interest rate is determined by inflation. By contrast, in the frictionless equilibrium, the real interest rate adjusts such as to equate aggregate demand to aggregate supply. Thus, in the absence of frictions, a rise in the supply of labor induces a fall in the real interest rate such as to generate a rise in the demand for consumption and for investment.[26]

[25] Recall that, by (9) and (15), we have $v'(l) = F_L(K, L)\, u'(c)$.

[26] It can be shown that in the frictionless equilibrium, if the real interest rate is negative (or positive but sufficiently close to zero), $c > \rho K$, and $-K\gamma''(K)/\gamma'(K) < -cu''(c)/u'(c)$, then a fall in the disutility of labor raises consumption, investment, and output. Also, if the real interest rate is negative (or positive but sufficiently close to zero), labor supply is sufficiently inelastic, and $-K\gamma''(K)/\gamma'(K) < -cu''(c)/u'(c)$, then a rise in total factor productivity raises output. Note that these sufficient conditions are far from necessary.

4 Policy Analysis

In the absence of government intervention, the economy is bound to remain permanently depressed. In this section, I therefore investigate the effectiveness of various stimulative policies. As the objective is to permanently escape secular stagnation, I focus on the long-run impact of these policies on the steady state equilibrium of the economy and abstract from transitional dynamics.

The obvious solution to secular stagnation would be to raise the inflation ceiling π^* to at least $\bar{\pi}$. However, recent history has shown that, for political and institutional reasons, this is typically not an option, even in crisis struck countries.

When the nominal interest rate is equal to zero, increasing the money supply through open market operations, i.e. by buying bonds, does not have any effect on the economy. Indeed, when $i = 0$, consumers' transaction demand for real money balances is satiated and money and bonds are perfect substitutes. Hence, any further increase in the money supply is used for savings: the economy is liquidity trapped!

An alternative policy option would be to implement a helicopter drop of money, i.e. to give households a fiscal transfer financed through an increase in the money supply. This policy violates the conservative version of the government's no-Ponzi condition (22).[27] Hence, it is only feasible if the government does not have to be able to redeem the outstanding money supply.

This policy increases both the wealth of households and the level of government liabilities. If households perceive the money supply as a government liability, then the helicopter drop of money does not increase their net wealth $a_t - (b_t^s + m_t^s)$ and, hence, cannot have any impact on the level of economic activity. If, on the contrary, households do not perceive the money supply as a government liability, then the policy raises their wealth, which reduces their marginal utility of wealth, and increases their demand for consumption, by the Euler equation (8) with $\dot{c}_t = 0$. If the transfers are sufficiently large to induce the economy to produce at full capacity, then the increase in the money supply must be highly inflationary, which is inconsistent with the low inflation ceiling. Thus, the helicopter drop of money does not seem to be a completely satisfactory solution to secular stagnation. Also, note that, in the special case where the marginal utility of wealth is constant, the policy is always completely ineffective.

The remaining conventional stabilization policy is to implement a fiscal stimulus. Let us therefore consider that the government relies on lump-sum taxes to finance a level g of government spending.[28] Aggregate demand becomes equal to $c + \delta K + g$. Note that a

[27] Let $(b_t, m_t)_{t=0}^{\infty}$ denote the pre-existing paths of real debt and money, which satisfy the conservative no-Ponzi condition $\lim_{t\to\infty} e^{-rt}(b_t + m_t) = 0$. The implementation of the helicopter drop of money at time 0 raises the initial real money supply by $\tilde{m}_0$. Hence, it increases the real government liabilities at time t by $\tilde{m}_t = \tilde{m}_0 P_0/P_t = \tilde{m}_0 e^{-\pi t} = \tilde{m}_0 e^{rt}$. We therefore have $\lim_{t\to\infty} e^{-rt}(b_t + m_t + \tilde{m}_t) = \tilde{m}_0 > 0$.

[28] If, alternatively, the government spending is financed through increases in the money supply, then

temporary increase in government expenditures cannot remedy secular stagnation. The increase in government spending must therefore be permanent.

Let $\varepsilon_u = -cu''(c)/u'(c)$ and $\varepsilon_\gamma = -K\gamma''(K)/\gamma'(K)$. In the presence of government spending, an exogenous rise in the real interest rate reduces both consumption and output if and only if $(c+g)\,\varepsilon_u > c\varepsilon_\gamma$, which is a generalization of Lemma 4. Throughout this section, I assume that this condition is satisfied.

Proposition 4 *In the secular stagnation equilibrium, in the limit as α tends to zero, i.e. when wages are completely rigid, the steady state government spending multiplier is equal to:*

$$1+\frac{\delta K + c\frac{\varepsilon_\gamma}{\varepsilon_u}}{(c+g)-c\frac{\varepsilon_\gamma}{\varepsilon_u}}. \tag{44}$$

If α is strictly positive, but sufficiently small, then government spending raises inflation, which generates an even larger multiplier.

In the limit as α tends to zero, nominal wages grow at rate π^R and the real interest rate is therefore equal to $-\pi^R$. In that case, government spending cannot affect the capital-labor ratio, by (16). Instead, a rise in g increases aggregate demand, which raises the demand for labor. But, to keep the capital-labor ratio constant, the capital stock must rise. So investment also rises. Hence, even when inflation does not respond to government spending, the multiplier is larger than 1. With a decreasing marginal utility of wealth, i.e. $\varepsilon_\gamma > 0$, this effect is amplified as the increase in the capital stock reduces the marginal utility of wealth, which raises the demand for consumption.

If wages are not completely rigid, then the increase in labor demand raises inflation. This spurs the demand for both consumption and investment, which further increases output.

However, a sizeable fiscal multiplier does not necessarily imply that the implementation of a fiscal stimulus is welfare enhancing.

Proposition 5 *In the secular stagnation equilibrium, in the limit as α tends to zero, i.e. when wages are completely rigid, a rise in government spending is welfare enhancing if and only if:*[29]

$$K\left[\rho+\pi^R\left(1-\frac{v'(L)}{v'(l)}\right)\right]+c\frac{\varepsilon_\gamma}{\varepsilon_u} > [c+g]\,\frac{v'(L)}{v'(l)}, \tag{45}$$

where $v'(L)/v'(l) \leq 1$ for all $L \leq l$.

the resulting policy can be seen as the joint implementation of the last two policies: a helicopter drop of money together with the fiscal stimulus financed from lump-sum taxes.

[29] This condition holds for any value of g, where c, l, L, and K implicitly are functions of g. It follows that the optimal value of g is such that the two sides of the inequality are equal to each other.

A rise in government spending affects welfare through three channels. First, to meet the higher level of demand, employment L rises. Agents work more, which reduces their welfare. Second, when wages are rigid, the capital-labor ratio is constant and the capital stock must therefore rise. This raises the wealth of households, net of government liabilities, which is welfare enhancing. Finally, if $\varepsilon_\gamma > 0$, the increase in wealth raises the demand for consumption, which is also welfare enhancing. However, even when $g = 0$, it is not clear whether the positive effects dominate.

If labor supply is completely inelastic, then $v'(L)/v'(l) = 0$ for all $L < l$. This implies that working more hours whenever $L < l$ is not costly. Hence, increasing g always raises welfare as long as $L < l$. This yields the following corollary.

Corollary 1 *If labor supply is completely inelastic, then the optimal fiscal stimulus eliminates the gap between labor demand L and labor supply l.*

It is important to realize that, even if $\varepsilon_\gamma = 0$, i.e. even if consumption is independent of g, eliminating under-employment is welfare enhancing as it raises the wealth of households while the disutility from supplying labor up to l is negligible.

If labor supply is infinitely elastic, then $v'(L)/v'(l) = 1$. In that case, under the mild condition that $c[1 - \varepsilon_\gamma/\varepsilon_u] > \rho K$, the implementation of any fiscal stimulus is detrimental to welfare, despite a multiplier that is larger than 1. By continuity, this result must also hold for a sufficiently high elasticity of labor supply.

Corollary 2 *If in the secular stagnation equilibrium $c[1 - \varepsilon_\gamma/\varepsilon_u] > \rho K$ and if labor supply is highly elastic, then the implementation of a fiscal stimulus reduces welfare.*

Thus, starting from a positive level of government spending, it would actually be desirable to cut government expenditures. This would allow workers to enjoy even more leisure.

Arguably, for plausible calibrations, $v'(L)/v'(l)$ is unlikely to be very close to zero, even when $g = 0$. For instance, a severe depression, where labor demand is only 75% of labor supply, together with a Frisch elasticity of labor supply of 0.5 implies that $v'(L)/v'(l) = (0.75)^{1/0.5} = 0.5625$. Of course, the case for a fiscal stimulus is stronger if wages are not completely sticky. However, the absence of outright deflation in almost any country during the Great Recession suggests that, in practice, inflation is not very responsive to government spending. All this suggests that, under secular stagnation, the case for a fiscal stimulus is, at best, weak.

My analysis of the fiscal stimulus so far assumes that the economy remains in the secular stagnation equilibrium. However, an increase in government spending raises the real interest rate of the frictionless equilibrium.[30] Thus, a fiscal stimulus that is sufficiently

[30] It can be shown that a sufficient condition for this is $c + g \geq \rho K$ and $f'(k) - \delta \leq 0$.

large to raise the frictionless interest rate from $-\bar{\pi}$ to $-\pi^*$ makes the frictionless equilibrium consistent with the inflation ceiling π^*. Note that, to implement this equilibrium, the money supply must be growing at rate π^*.

Of course, this policy does not eliminate the secular stagnation equilibrium. The difficulty is to make sure that the economy converges to the frictionless equilibrium. If this does not spontaneously occur, the government can eliminate the secular stagnation equilibrium by committing to prevent inflation from falling below π^R. Indeed, at any point in time, it can always spend sufficiently to eliminate the gap between labor demand L and labor supply l, which raises inflation to at least π^R, by (36). In theory, if this commitment is fully credible, households expect the economy to converge to the frictionless steady state with inflation π^*. In practice, it might be very difficult for a government to credibly commit to spend sufficiently to prevent the occurrence of a steady state with under-employed workers.[31]

5 Optimal Policy

To lift the economy out of secular stagnation, we clearly need to look beyond the usual set of policy instruments used for macroeconomic stabilization. Hence, I now characterize the optimal policy under a much richer set of taxes and subsidies. I proceed in two steps. First, I determine the optimal allocation of resources. I then solve for the fiscal policy that implements this allocation in the decentralized economy.

5.1 Optimal Allocation

Wealth can either be held in the form of physical capital K_t, bonds b_t, or money m_t. Hence, in equilibrium, the private wealth of households net of government liabilities must be equal to K_t. The planner's problem is to maximize the welfare of the representative household:

$$\int_0^\infty e^{-\rho t}\left[u\left(c_t\right)-v\left(l_t\right)+h\left(m_t\right)+\gamma\left(K_t\right)\right]dt, \tag{46}$$

subject to the resource constraint given by the capital accumulation equation:

$$\dot{K}_t = F\left(K_t, l_t\right) - \delta K_t - c_t, \tag{47}$$

where I have used the fact that, in a first-best allocation, the labor used in production L_t must be equal to the labor supplied by workers l_t. By the maximum principle, for any

[31] In the late 1930s, Hansen (1939) believed that the U.S. economy was trapped into a secular stagnation equilibrium. A possible interpretation of the subsequent history is that the military spending of World War II allowed the economy to escape secular stagnation, while the build-up of the welfare state after WWII may have been sufficient to allow the economy to produce at full capacity.

initial stock K_0 of capital, the first-best allocation $(c_t^*, K_t^*, l_t^*, m_t^*)$ is characterized by:[32]

$$\dot{K}_t^* = F\left(K_t^*, l_t^*\right) - \delta K_t^* - c_t^*, \tag{48}$$

$$\frac{\dot{c}_t^*}{c_t^*} = \left[F_K\left(K_t^*, l_t^*\right) - \delta - \rho + \frac{\gamma'\left(K_t^*\right)}{u'\left(c^*\right)}\right] \frac{u'\left(c_t^*\right)}{-u''\left(c_t^*\right) c_t^*}, \tag{49}$$

$$v'\left(l_t^*\right) = F_L\left(K_t^*, l_t^*\right) u'\left(c_t^*\right), \tag{50}$$

$$h'\left(m_t^*\right) = 0, \tag{51}$$

together with the transversality condition:

$$\lim_{t\to\infty} e^{-\rho t} u'\left(c_t^*\right) K_t^* = 0. \tag{52}$$

Money has no impact on the allocation of resources, but is valued by consumers who derive utility from holding money. Hence, the first-best allocation satiates individuals with real money balances.

These equations correspond to the frictionless equilibrium of the decentralized economy where consumption is given by (8) with $r_t = F_K\left(K_t, l_t\right) - \delta$ and $a_t - (b_t + m_t) = K_t$, labor supply by (9) with $w_t = F_L\left(K_t, l_t\right)$, and where aggregate supply $F\left(K_t, l_t\right)$ is equal to aggregate demand $c_t + \delta K_t + \dot{K}_t$. Also, given the money demand equation (10), to have $h'\left(m_t\right) = 0$ in a decentralized economy, we need to have $i_t = 0$, i.e. the Friedman (1969) rule needs to hold. This establishes the following result.

Proposition 6 *The frictionless equilibrium with $i_t = 0$ does implement the first-best allocation of resources.*

We have previously established that, under mild conditions, the frictionless steady state is unique and stable. We can therefore consider that the first-best allocation converges to that steady state.

Let r_t^n denote the natural real interest rate of the frictionless equilibrium, defined as:

$$r_t^n = F_K\left(K_t^*, l_t^*\right) - \delta. \tag{53}$$

Recall that, to have secular stagnation with $\pi^* \geq 0$, the steady state natural real interest rate must be negative, i.e. $r^n = -\bar{\pi} < -\pi^* \leq 0$. This implies that, in the first-best steady state, the capital stock must be above the golden rule level. However, this is fully efficient as households derive utility from holding real wealth.

[32] It is straightforward to check that the Hamiltonian is strictly concave in c_t, l_t, m_t, and K_t and that the corresponding current-value costate variable is equal to $u'\left(c_t\right)$ and that it is therefore always strictly positive. Hence (by Acemoglu 2009, Theorem 7.14, page 257), for a given value of K_0, the optimality conditions define the unique solution to the optimal control problem.

5.2 Implementation in the Decentralized Economy

For simplicity, firms are assumed to rent capital owned by households. As investment is always positive in the first-best allocation that I want to implement, I assume that the price of capital is always equal to that of consumption goods, i.e. $p_t^K = P_t^K/P_t = 1$. A capital income tax τ_t^K is imposed on the income from physical capital $R_t K_t$ net of a depreciation allowance $\varphi_t \delta P_t K_t$, where, as before, R_t denotes the nominal rental cost of capital.[33] Under full depreciation allowance, we have $\varphi_t = 1$. There is an investment subsidy s_t^I. As we shall see, this reduces the effective price of capital at time t to $P_t\left(1 - s_t^I\right)$. The effective nominal wealth of a household is therefore equal to:

$$A_t = \left(1 - s_t^I\right) P_t K_t + B_t + M_t. \tag{54}$$

This wealth is taxed at rate τ_t^W at each point in time.[34] Finally, households are subject to a labor income tax τ_t^L and to a consumption tax τ_t^C.

This fiscal policy implies that the nominal wealth $P_t K_t + B_t + M_t$ of the representative household evolves according to:

$$\begin{aligned} & P_t\dot{K}_t + \dot{P}_t K_t + \dot{B}_t + \dot{M}_t \\ &\quad = i_t B_t + \left(R_t K_t - \tau_t^K \left[R_t K_t - \varphi_t \delta P_t K_t\right] + s_t^I \left[P_t \dot{K}_t + \delta P_t K_t\right] - \delta P_t K_t + \dot{P}_t K_t\right) \\ &\qquad -\tau_t^W \left[\left(1 - s_t^I\right) P_t K_t + B_t + M_t\right] + \left(1 - \tau_t^L\right) W_t l_t + P_t \psi_t - \left(1 + \tau_t^C\right) P_t c_t, \end{aligned} \tag{55}$$

where nominal investment is equal to $P_t\dot{K}_t + \delta P_t K_t$. Using the fact that $\dot{B}_t/P_t = \dot{b}_t + \pi_t b_t$ and $\dot{M}_t/P_t = \dot{m}_t + \pi_t m_t$, this can be written in real terms as:

$$\begin{aligned} \left(1 - s_t^I\right)\dot{K}_t - \dot{s}_t^I K_t + \dot{b}_t + \dot{m}_t &= \left(i_t - \pi_t - \tau_t^W\right)\left[\left(1 - s_t^I\right) K_t + b_t + m_t\right] \\ &\quad + \left[\left(1 - \tau_t^K\right) R_t/P_t + \tau_t^K \varphi_t \delta - \left(1 - s_t^I\right)\delta - \dot{s}_t^I - \left(1 - s_t^I\right)\left(i_t - \pi_t\right)\right] K_t \\ &\quad - i_t m_t + \left(1 - \tau_t^L\right) w_t l_t + \psi_t - \left(1 + \tau_t^C\right) c_t. \end{aligned} \tag{56}$$

This expression does confirm that the effective real price of capital is $\left(1 - s_t^I\right)$. By arbitrage, the returns from holding capital should be identical to the returns from holding bonds. Thus, the second term on the right hand side of (56) must be equal to zero, which yields the following relationship between the real interest rate and the user cost of capital:

$$r_t = \frac{\left(1 - \tau_t^K\right) R_t/P_t + \tau_t^K \varphi_t \delta - \dot{s}_t^I}{1 - s_t^I} - \delta. \tag{57}$$

[33] If firms were the owners of capital, then τ_t^K would correspond to a corporate income tax.

[34] In the optimal allocation, the nominal return from wealth is equal to zero and the real return is negative (as, in steady state, $r^n < -\pi^* \leq 0$). Hence, in that context, it seems more natural to tax the stock of wealth rather than the (negative) income from wealth.

The wealth accumulation equation simplifies to:

$$\dot{a}_t = \left(r_t - \tau_t^W\right) a_t - i_t m_t + \left(1 - \tau_t^L\right) w_t l_t + \psi_t - \left(1 + \tau_t^C\right) c_t, \tag{58}$$

where $a_t = A_t/P_t$. The corresponding no-Ponzi condition is:

$$\lim_{t\to\infty} e^{-\int_0^t \left(r_s - \tau_s^W\right) ds} a_t \geq 0. \tag{59}$$

Finally, households' intertemporal utility is given by:

$$\int_0^\infty e^{-\rho t} \left[u\left(c_t\right) - v\left(l_t\right) + h\left(\frac{m_t}{1+\tau_t^C}\right) + \gamma\left(\frac{a_t - \left(b_t^s + m_t^s - s_t^I K_t\right)}{1+\tau_t^C}\right)\right] dt, \tag{60}$$

where I assume that they care about the value of money and wealth at consumer prices, including consumption taxes. The sum of the wealth of the private and of the public sector must always be equal to the stock of physical capital. Hence, the government liabilities at t are equal to $b_t^s + m_t^s - s_t^I K_t$, where b_t^s, m_t^s, and K_t are aggregate quantities that are outside the control of any single household.

The consumer's problem is to maximize utility, (60), subject to the budget constraint, (58) and (59), with a_0 given. The solution to the problem is characterized by:

$$\frac{\dot{c}_t}{c_t} = \left[r_t - \tau_t^W - \frac{\dot{\tau}_t^C}{1+\tau_t^C} - \rho + \frac{1}{u'\left(c_t\right)} \gamma' \left(\frac{a_t - \left(b_t^s + m_t^s - s_t^I K_t\right)}{1+\tau_t^C}\right)\right] \frac{u'\left(c_t\right)}{-u''\left(c_t\right) c_t}, \tag{61}$$

$$v'\left(l_t\right) = \frac{1-\tau_t^L}{1+\tau_t^C} w_t u'\left(c_t\right), \tag{62}$$

$$h'\left(\frac{m_t}{1+\tau_t^C}\right) = i_t u'\left(c_t\right), \tag{63}$$

together with the transversality condition:

$$\lim_{t\to\infty} e^{-\rho t} \frac{u'\left(c_t\right)}{1+\tau_t^C} a_t = 0. \tag{64}$$

The reason why $\dot{\tau}_t^C$ appears in the Euler equation (61) is that a rising path of the consumption tax makes future consumption expansive relative to current consumption. This reduces the effective real interest rate faced by consumers, which raises the demand for consumption.

On the supply side, the constant returns to scale production function implies that, in equilibrium, the real rental cost of capital and the real wage are given by:

$$\frac{R_t}{P_t} = F_K\left(K_t, L_t\right), \tag{65}$$

$$w_t = F_L\left(K_t, L_t\right), \tag{66}$$

respectively.

The downward wage rigidity is the same as before, summarized by (36). As the first-best allocation is characterized by full employment, i.e. $L_t = l_t$, it cannot be implemented with inflation below π^R. Thus, as we shall see, the first-best allocation can be implemented for any given path of inflation such that $\pi_t \in \left[\pi^R, \pi^*\right]$. This requires the money supply to grow at rate π_t at time t.

In the first-best allocation, households have a zero marginal utility of real money balances, by (51). The decentralized money demand equation (63) therefore implies that, to implement the optimal allocation, we must have $i_t = 0$ and, hence, $r_t = -\pi_t$.

Fundamentally, in the laissez-faire economy, the lower bound on the real interest rate, $r_t \geq -\pi^*$, creates two distortions that need to be corrected. First, by (8), it reduces the demand for consumption. Second, by (16), it reduces the capital-labor ratio.

To correct the first distortion, the net-of-tax interest rate $r_t - \tau_t^W - \dot{\tau}_t^C / \left(1 + \tau_t^C\right) = -\pi_t - \tau_t^W - \dot{\tau}_t^C / \left(1 + \tau_t^C\right)$ in the demand for consumption (61) must be equal to the natural real interest rate r_t^n. As in practice the consumption tax cannot be increasing forever, I impose that it must be constant over time, i.e. $\dot{\tau}_t^C = 0$. To implement the first-best allocation for a given $\pi_t \in \left[\pi^R, \pi^*\right]$, the wealth tax must therefore be given by:

$$\tau_t^W = -r_t^n - \pi_t. \tag{67}$$

This guarantees that the consumption Euler equation of the decentralized economy (61) coincides with the corresponding first-order condition to the planner's problem (49). Secular stagnation results from the fact that in steady state $r_t^n = r^n < -\pi^*$, which implies that $\tau_t^W = -r^n - \pi_t > \pi^* - \pi_t \geq 0$.

To correct the second distortion, the marginal product of capital of the frictionless equilibrium $F_K\left(K_t^*, l_t^*\right) = \delta + r_t^n$ must be made consistent with real interest rate of the decentralized economy $r_t = -\pi_t$. Hence, by the arbitrage relationship (57), the capital income tax τ_t^K, the depreciation allowance φ_t, and the investment subsidy s_t^I must be jointly chosen such that:

$$-\pi_t = \frac{\left(1 - \tau_t^K\right)\left[\delta + r_t^n\right] + \tau_t^K \varphi_t \delta - \dot{s}_t^I}{1 - s_t^I} - \delta. \tag{68}$$

Let us consider three special cases of interest. For simplicity, I focus on the optimal policy with constant inflation $\pi \in \left[\pi^R, \pi^*\right]$ and in steady state, where $r_t^n = r^n = -\bar{\pi}$. First, relying exclusively on the investment subsidy, i.e. $\tau^K = 0$, we have:

$$s^I = \frac{-r^n - \pi}{\delta - \pi}, \tag{69}$$

which is strictly positive, as $r^n < -\pi^* \leq -\pi$ and $\delta - \pi > \delta - \bar{\pi} = \delta + r^n = F_K(K^*, l^*) > 0$. Second, relying on the capital income tax with full depreciation allowance, i.e. $s^I = 0$ and $\varphi = 1$, we obtain:

$$\tau^K = \frac{-r^n - \pi}{-r^n}, \tag{70}$$

which is strictly positive (to the extent that $\pi^* \geq 0$ implies $r^n < 0$). In fact, the negative natural real interest rate implies that the returns from capital in the frictionless equilibrium are negative. Hence, with full depreciation allowance, a positive capital income tax absorbs some of the losses from capital. It therefore subsidizes capital. Third, relying again on the capital income tax but with no depreciation allowance, i.e. $s^I = 0$ and $\varphi = 0$, we have:

$$\tau^K = \frac{r^n + \pi}{\delta + r^n}, \tag{71}$$

which, in steady state, is negative as $r^n + \pi = -\bar{\pi} + \pi < 0$. With no depreciation allowance, the capital income tax needs to be negative to subsidize capital. Importantly, these three policies are equivalent as they all implement the same marginal product of capital $r^n + \delta = F_K(K^*, l^*)$.

The optimality condition for labor supply (50) implies that, if both consumption and the capital-labor ratio are at their first-best levels, then so is the supply of labor. Hence, by (62) and (66), the labor income and consumption taxes can both be set equal to zero, i.e. $\tau_t^L = \tau_t^C = 0$.

Let us now determine the level of the lump-sum transfers ψ_t that balances the intertemporal government budget constraint. The accumulation of government liabilities is given by:

$$\dot{b}_t^s + \dot{m}_t^s = r_t \left[b_t^s + m_t^s\right] - i_t m_t^s + \psi_t + s_t^I \left[\dot{K}_t + \delta K_t\right] - \tau_t^K \left[R_t/P_t - \varphi_t \delta\right] K_t - \tau_t^W a_t. \tag{72}$$

Rearranging terms, this can be written as:

$$\begin{aligned} \dot{b}_t^s + \dot{m}_t^s - s_t^I \dot{K}_t - \dot{s}_t^I K_t = {} & \left(r_t - \tau_t^W\right) \left[b_t^s + m_t^s - s_t^I K_t\right] \\ & - i_t m_t^s + \psi_t + s_t^I \left[r_t + \delta\right] K_t - \dot{s}_t^I K_t - \tau_t^K \left[R_t/P_t - \varphi_t \delta\right] K_t - \tau_t^W K_t. \end{aligned} \tag{73}$$

But, by the arbitrage relationship (57), we know that $s_t^I \left[r_t + \delta\right] - \dot{s}_t^I - \tau_t^K \left[R_t/P_t - \varphi_t \delta\right] = r_t + \delta - R_t/P_t$. As the sum of the wealth of the private and of the public sector must be equal to the stock of physical capital, the government is always solvent, and able to redeem the money supply, provided that it satisfies the following no-Ponzi condition:[35]

$$\lim_{t\to\infty} e^{-\int_0^t \left(r_s-\tau_s^W\right)ds}\left[b_t^s+m_t^s-s_t^I K_t\right]\leq 0. \tag{74}$$

Hence, the government's intertemporal budget constraint is given by:

$$\begin{aligned} & b_0+m_0-s_0^I K_0 \\ & \quad +\int_0^\infty e^{-\int_0^t\left(r_x-\tau_x^W\right)dx}\left[-i_t m_t^s+\psi_t+\left(r_t+\delta-R_t/P_t\right)K_t-\tau_t^W K_t\right]dt\leq 0, \end{aligned} \tag{75}$$

But, in the first-best allocation, $i_t = 0$. Also, the optimal policy for a given $\pi_t \in \left[\pi^R, \pi^*\right]$ is such that $r_t = -\pi_t$, $R_t/P_t - \delta = r_t^n$ and $\tau_t^W = -r_t^n - \pi_t$. It immediately follows that the government budget constraint simplifies to:

$$b_0+m_0-s_0^I K_0+\int_0^\infty e^{-\int_0^t r_x^n dx}\psi_t dt\leq 0. \tag{76}$$

This shows that, if $s_0^I = 0$, the optimal policy is self-financing as the wealth tax pays for the subsidization of income from physical capital. The only effect of the policy is to modify the real interest rate at which future lump-sum transfers are discounted.

The initial investment subsidy s_0^I reduces the initial price of capital, which tightens the budget constraint of households. This mechanically relaxes the government budget constraint. Hence, s_0^I acts as a levy on initial capital. However, if the natural real interest rate r_t^n is eventually negative, then infinitesimally small negative lump-sum transfers ψ_t forever are sufficient to balance the government budget constraint for any finite value of s_0^I. In that case, the lump-sum transfers that the government can afford are not affected by s_0^I.

Proceeding as before, cf. (41), it is straightforward to check that, under the optimal policy, the transversality conditions of the household (64) and of the planner (52) coincide.

We have therefore characterized by construction a policy that can implement the first-best allocation for any path of inflation such that $\pi_t \in \left[\pi^R, \pi^*\right]$ at all time t. The main result is summarized in the following proposition.

Proposition 7 *At any time T, the first-best allocation can be implemented in the decentralized economy for any given path of inflation $(\hat{\pi}_t)_{t=T}^{\infty}$ such that $\hat{\pi}_t \in \left[\pi^R, \pi^*\right]$ for all $t \geq T$ by setting the money growth rate ϕ_t, the tax on wealth τ_t^W, the capital income tax*

[35] This is a generalization of the conservative government budget constraint, given by (22).

τ_t^K, the depreciation allowance φ_t, and the investment subsidy s_t^I such that:

$$\phi_t = \hat{\pi}_t, \tag{77}$$
$$\tau_t^W = -r_t^n - \hat{\pi}_t, \tag{78}$$
$$\left(1 - s_t^I\right)[\delta - \hat{\pi}_t] = \left(1 - \tau_t^K\right)[\delta + r_t^n] + \tau_t^K \varphi_t \delta - \dot{s}_t^I. \tag{79}$$

This policy is revenue neutral, except that s_0^I acts as a levy on initial capital.

The optimal policy implements the allocation of the frictionless equilibrium with $i_t = 0$. However, it implements this allocation with inflation $\pi_t = \hat{\pi}_t \in \left[\pi^R, \pi^*\right]$, rather than the much higher rate $\pi_t = -r_t^n$.

By construction, the optimal policy is consistent with the inflation ceiling π^* at all time, except possibly when the policy is implemented. Indeed, as the economy jumps from the secular stagnation to the frictionless equilibrium, the zero lower bound ceases to bind.[36] If the policy is implemented at time T with $i_t = 0$, then the price level at T must be given by:

$$P_T = \frac{M_T}{\bar{m}}.$$

To prevent the price level from jumping upon implementation of the policy, the money supply at time T must be equal to $\bar{m}P_{T-\varepsilon}$, where $P_{T-\varepsilon}$ denotes the price level that prevailed just before. If the real money supply at time $T-\varepsilon$ was larger than the transaction demand for money, i.e. $m_{T-\varepsilon} > \bar{m}$, the government must redeem a quantity $m_{T-\varepsilon} - \bar{m}$ of money. This can generate a massive increase in government debt.[37] However, in theory, with a permanently negative natural real interest rate, the government's no-Ponzi condition (74) is satisfied for an arbitrarily large initial level of debt, provided that the primary budget is subsequently balanced. These considerations nevertheless show that, if the government used to rely excessively on the printing press to finance its expenditures, then the sustainability of public debt can become an important issue when the optimal policy is implemented.

For simplicity, let us consider that the government targets a constant rate of inflation $\hat{\pi} \in \left[\pi^R, \pi^*\right]$ and that, once the economy is in steady state, the policy becomes time

[36] Even if the nominal interest rate remains equal to zero, the zero lower bound is no longer preventing that rate from being negative. Hence, as explained in the derivation of the steady state equilibrium, an equilibrium where some money is used for saving is unstable.

[37] In my model, the government corresponds to the consolidation of the ministry of finance and of the central bank. In practice, the central bank can redeem the money supply by selling the government bonds that it previously acquired to finance the public deficit. This does not change the indebtedness of the ministry of finance, but raises the amount of debt held by the private sector rather than by the central bank. In Japan, the debt-to-GDP ratio amounts to 230% of GDP, a quarter of which is held by the central bank. Thus, the amount of debt held by the private sector only amounts to 170% of GDP. If Japan escapes the liquidity trap, this ratio would need to increase sharply to avoid a jump in the price level.

invariant, i.e. $\phi_t = \phi$, $\tau_t^W = \tau^W$, $\tau_t^K = \tau^K$, $\varphi_t = \varphi$, and $s_t^I = s^I$. Importantly, the optimal policy does not eliminate the secular stagnation equilibrium, where inflation is below the target $\hat{\pi}$.[38] There is however one important exception.

Proposition 8 *The frictionless and secular stagnation steady state equilibria can coincide if and only if the optimal policy is implemented with inflation target $\hat{\pi} = \pi^R$. If the secular stagnation equilibrium is unique, which must be the case for α sufficiently close to zero, then the optimal policy with $\hat{\pi} = \pi^R$ uniquely implements the first-best allocation.*

Implementing the optimal allocation with inflation target $\hat{\pi} = \pi^R$ requires the most aggressive policy intervention, i.e. the largest wealth tax and the highest subsidization of physical capital. Indeed, a massive intervention is necessary to make the optimal allocation consistent with an inflation rate as low as π^R.

If the government would like to be in the optimal allocation with inflation π^*, then it should first implement the optimal policy with $\hat{\pi} = \pi^R$ in order to escape secular stagnation. It should then raise the growth rate of the money supply to π^*, while reducing the size of the wealth tax, and of the subsidization of capital that it finances, until the nominal interest rate is back to zero.

The wealth of households is composed of physical capital, bonds, and money. A potential concern with the optimal policy is that money might be harder to tax than physical capital or bonds. Households might indeed be tempted to accumulate stockpiles of cash to avoid paying the wealth tax. But, untaxed money yields a real interest rate $-\pi_t$, which, under the inflation ceiling, is higher than the steady state natural real interest rate. By arbitrage, this would induce the after-tax real interest rate on physical capital and on bonds to be above the natural rate, which would prevent the implementation of the optimal allocation. It is therefore essential that the government is able to tax money.

To fight against money laundering, many countries already enforce significant restrictions that reduce the value of large sums of undeclared currency. There are limits to the size of payments than can be made in cash.[39] Banks cannot accept large cash deposits if the origin of the funds is not proven. Controls can also be made when old bills occasionally become obsolete and are replaced by new ones. If these measures are not sufficient, then the government can adopt a payment system that exclusively relies on electronic money.

Note that, if money is straightforward to tax, then an alternative way to implement the first-best allocation is to alleviate the zero lower bound on the nominal interest rate. For this, a government targeting inflation $\hat{\pi}_t$ only needs to tax money at rate τ_t^W, while

[38] Note that, the optimal policy is revenue neutral even if it fails to implement the optimal allocation.

[39] In France, the limit is 1 000 euros for commercial transactions and 10 000 euros for real estate transactions.

setting the money growth rate ϕ_t and the tax rate on money τ_t^W according to (77) and (78), respectively. The lower bound on the nominal interest rate becomes $-\tau_t^W$, which is negative. In that case, when the nominal rate is at its lower bound, the real interest rate is at its natural level, as $i_t = -\tau_t^W = r_t^n + \hat{\pi}_t$ implies $r_t = i_t - \hat{\pi}_t = r_t^n$. This policy is revenue neutral.

In the absence of electronic money, the tax on wealth seems to offer a more reliable way to reduce the net-of-tax real interest rate faced by households than the tax on money. Indeed, the wealth tax is effective provided that the authorities can prevent large accumulations of undeclared currency. By contrast, if real money balances are small and the tax on money is poorly enforced on small amounts, then it is not clear that, by arbitrage, it can reduce the net-of-tax real interest rate faced by households on their holdings of physical capital and bonds. This suggests that, without electronic money, it is likely to be very difficult to enforce negative nominal interest rates.

6 Heterogeneity

For simplicity, I have so far relied on a representative household model. However, my results only require a strictly positive mass of households to have a preference for wealth. Indeed, in the presence of heterogeneity, impatient households borrow from those who are more patient, until they reach their borrowing limit. Once this limit is reached, the real interest rate must be sufficiently low to induce the most patient households to consume enough to allow the economy to produce at full capacity. In other words, in steady state, the natural real interest rate must be determined by the behavior of the most patient households, i.e. by those who have a preference for wealth.

Under a dynastic interpretation of the infinitely lived households, the borrowing limit should be determined by the impossibility of bequeathing negative wealth. For instance, in a discrete time model where one period corresponds to one generation, households face a zero debt limit. As the borrowing limit must eventually be binding, I consider a very simple form of heterogeneity whereby a fraction of households have a preference for wealth, while the remaining fraction consists of hand-to-mouth consumers who do not derive any utility from holding wealth.

Interestingly, in this setup, if the real interest rate is excessively high, then the economy can only produce at full capacity if poor households accumulate an excessive amount of debt. But, once the rich lenders realize that debt cannot be passed on to the next generation of poor households, the economy has to go through a painful deleveraging episode. As pointed out by Summers (2014), the fact that the credit boom that occurred before the Great Recession did not overheat the U.S. economy is fully consistent with the secular stagnation hypothesis.

In the presence of heterogeneity, the government might want to redistribute resources across households. In general, as in Correia, Farhi, Nicolini and Teles (2013), the set of implementable allocation is not modified by the zero lower bound. Indeed, the government can always implement its desired redistribution policy, while using wealth taxes and investment subsidies as described in the previous section to overcome the zero lower bound.[40]

In the context of secular stagnation, there is however one important exception where the zero lower bound can allow the government to implement an allocation that would not otherwise be feasible. This occurs when the government would like to redistribute to hand-to-mouth households, but cannot rely on household-specific lump-sum taxes or subsidies and does not want to modify the tax rate on wealth, as this would distort the consumption behavior of wealthy households.[41] Hence, the government can only raise the consumption of the hand-to-mouth by giving a lump-sum subsidy ψ_t to all households. If ψ_t is large, this clearly violates the conservative government budget constraint, given by:

$$b_0 + m_0 + \int_0^\infty e^{-\int_0^t r_x dx} \left[\psi_t - i_t m_t\right] dt \leq 0. \tag{80}$$

But, if the government does not need to redeem the outstanding money supply, the budget constraint becomes:

$$b_0 + \int_0^\infty e^{-\int_0^t r_x dx} \left[\psi_t - \phi_t m_t\right] dt \leq 0. \tag{81}$$

The subsidy can be financed by printing money forever. The wealth of rich households must therefore grow without bounds. Indeed, if the conservative version of the government's no-Ponzi condition is violated and if markets clear, then, by Walras' law, the no-Ponzi condition of rich households must be slack, for any value of the real interest rate. The increase in wealth is not distortionary, i.e. it does not affect the behavior of rich households, provided that it does not change their marginal utility of wealth. This either occurs if they consider money as a government liability, or if they have a constant marginal utility of wealth.

Thus, in the absence of household specific lump-sum taxes, the only way to redistribute to the poor without increasing the tax rate on wealth is to print money. This policy can be sustained forever provided that the wealthy have a zero marginal propensity to consume out of extra money supplied.[42] Paradoxically, this egalitarian allocation of consumption can only be implemented by making the rich infinitely wealthy.

[40] Under secular stagnation, it might be tempting to stimulate aggregate demand by taxing the wealthy and giving lump-sum subsidies to hand-to-mouth households. However, if this is not optimal in the absence of the zero lower bound, then it remains suboptimal with it.

[41] Given the limited amount of heterogeneity in the model, a non-linear wealth tax could act as a lump-sum tax. I therefore assume that this instrument is not available.

[42] This assumption does not seem outrageous in light of the recent Japanese experience, where a 230% debt-to-GDP ratio fails to lift the economy out of secular stagnation.

This theoretical possibility is specific to secular stagnation where the zero lower bound is potentially binding forever. Needless to emphasize that such policy is very dangerous should the economy ever escape secular stagnation. Hence, crucially, the increase in the consumption of the hand-to-mouth should not induce the economy to produce at full capacity, as inflation would then explode.

7 Conclusion

In this paper, I have relied on a Ramsey model with money to offer a simple theory of secular stagnation. The permanent failure of the economy to produce at full capacity results from three plausible assumptions: (i) the central bank has a low inflation ceiling; (ii) a positive mass of households has a very high propensity to save, due to a preference for wealth; (iii) wages are downward rigid. Despite the simplicity of this theory, I was able to derive the main Keynesian paradoxes.

I have argued that traditional monetary and fiscal policies are not efficient in the context of a permanent liquidity trap. To escape secular stagnation without raising the inflation ceiling, the government should instead simultaneously tax wealth and subsidize income from physical capital, either through an investment subsidy or a reduction in the taxation of corporate income. This policy is revenue neutral and implements the first-best allocation of resources.

When the economy escapes the liquidity trap, households cease to rely on money for saving. To prevent the price level from jumping upon implementation of the optimal policy, the government must therefore shrink the money supply to the level of the transaction demand for money. This can significantly raise the debt-to-GDP ratio. In theory, an arbitrarily large amount of public debt is sustainable when the real interest rate is negative forever. But, in practice, a country that used to heavily rely on monetary financing of public deficits might be reluctant to implement the optimal policy. Hence, an excessive reliance on the printing press can eventually result in a country choosing to remain permanently trapped into secular stagnation.

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A Proof of Lemma 1

Let $k = K/L$ and $f(k) = F(k,1)$. This implies that $F_K(K,L) = f'(k)$ and $F_L(K,L) = f(k) - kf'(k)$. Note that $f(0) - 0f'(0) = F_L(0,L) = 0$ as $F(0,L) = 0$ for any value of L.

In a frictionless equilibrium with $\pi \geq \bar{\pi} > \pi^R$, the labor market must clear, i.e. $L = l$. The goods market clearing condition (24) implies that:

$$l\left[f(k) - \delta k\right] = c. \tag{A1}$$

By (9) and (15), we have:

$$v'(l) = \left[f(k) - kf'(k)\right] u'(c). \tag{A2}$$

Finally, by (8) with $\dot{c}_t = 0$ and $a_t - (b_t + m_t) = K = kl$ and by (16), we have:

$$u'(c) = \frac{\gamma'(kl)}{\rho - \left[f'(k) - \delta\right]}. \tag{A3}$$

These three equations jointly characterize the steady state frictionless equilibrium.

Let $\hat{k}$ be uniquely defined by $f(\hat{k}) = \delta\hat{k}$. Substituting (A1) into (A2) yields:

$$v'\left(\frac{c}{f(k) - \delta k}\right) = [f(k) - kf'(k)]\, u'(c)\,. \tag{A4}$$

For any given value of $k \in (0, \hat{k})$, as c increases from 0 to $+\infty$, the right hand side decreases from $+\infty$ to 0 while the left hand side is increasing and non-negative. Hence, for any value of $k \in (0, \hat{k})$, there is a unique corresponding value of c. Moreover, differentiating (A4) with respect to k reveals that c is a continuous function of k for $k \in (0, \hat{k})$. Also, in the limit as k tends to $\hat{k}$, c must tend to 0.[43]

Let $\tilde{k}$ be uniquely defined by $\rho = f'(\tilde{k}) - \delta$. Clearly, $f'(\hat{k}) - \delta < 0 < \rho = f'(\tilde{k}) - \delta$. Hence, $\tilde{k} < \hat{k}$. Substituting (A1) into (A3) yields:

$$\gamma'\left(\frac{kc}{f(k) - \delta k}\right) = [\rho + \delta - f'(k)]\, u'(c)\,. \tag{A5}$$

Note that $k/[f(k) - \delta k]$ is an increasing function of k.[44] Let $\bar{c}$ be defined by $[\rho + \delta - f'(\hat{k})]u'(\bar{c}) = \lim_{K\to+\infty} \gamma'(K)$. Thus, as k tends to $\hat{k}$ from below, c tends to $\bar{c}$.[45] For any given value of $c \in (0, \bar{c})$, as k increases from $\tilde{k}$ to $\hat{k}$, the right hand side of (A5) increases from 0 to $[\rho + \delta - f'(\hat{k})]u'(c) > [\rho + \delta - f'(\hat{k})]u'(\bar{c})$ while the left hand side is weakly decreasing and tends to $\lim_{K\to+\infty} \gamma'(K) = [\rho + \delta - f'(\hat{k})]u'(\bar{c})$ as k tends to $\hat{k}$. Hence, for any value of $c \in (0, \bar{c})$, there is a unique corresponding value of $k \in (\tilde{k}, \hat{k})$. Moreover, differentiating (A5) with respect to c reveals that k is a continuous function of c for $c \in (0, \bar{c})$.

The steady state frictionless equilibrium is fully characterized by (A4) and (A5). But, by (A4), c is a continuous function of k for $k \in (0, \hat{k})$, which tends to 0 as k tends to $\hat{k}$; while, by (A5), k is a continuous function of c for $c \in (0, \bar{c})$ with $k \in (\tilde{k}, \hat{k})$. Hence, there exists at least one value of $k \in (\tilde{k}, \hat{k})$ that corresponds to a frictionless equilibrium.

If $\rho = 0$, then we have $f'(\tilde{k}) - \delta = 0$. If labor supply is inelastic, then, by (A1), c is a strictly decreasing function of k for all $k \in (\tilde{k}, \hat{k})$, while, by (A3), c is a strictly increasing function of k for all $k \in (\tilde{k}, \hat{k})$. Hence, the equilibrium value of $k \in (\tilde{k}, \hat{k})$ must be unique. By continuity, the results also holds provided that ρ is sufficiently small and that labor supply is sufficiently inelastic.

[43] If the maximum feasible labor supply $\bar{l}$ is finite, then, by (A1), c must tend to 0 while, by (A2), l must tend to $\bar{l}$, as $\lim_{l\to\bar{l}} v'(l) = \infty$. If $\bar{l}$ is infinite, then c cannot tend to a strictly positive value, as, by (A2), l would also tend to a finite value which would be inconsistent with (A1). Hence, c must tend to 0 while l must tend to ∞.

[44] Its derivative is equal to $[f(k) - kf'(k)] / [f(k) - \delta k]^2$, which is positive as $f(k) - kf'(k) = F_L(K, L)$.

[45] Note that c cannot tend to 0 as $\lim_{K\to 0} \gamma'(K) < \infty$.

B Proof of Lemma 2

As in the proof of the previous lemma, let $k = K/L$ and $f(k) = F(k, 1)$. To prove the lemma, we compare the slope of the aggregate demand and of the aggregate supply curves. For this, we first need to compute the derivative of c, k, and l with respect to r, where $r = i - \pi$ with $i \geq 0$. Recall that, when $\pi \in \left[\pi^R, \bar{\pi}\right)$, the labor market clears, i.e. $L = l$.

From the equation for the demand for capital (16), we have:

$$\frac{dk}{dr} = \frac{1}{f''(k)} < 0.$$

Implicitly differentiating the optimality condition for labor supply (9) with $w = f(k) - kf'(k)$ yields:

$$\begin{aligned}
\frac{dl}{dr} &= \frac{1}{v''(l)}\left[-kf''(k)\,u'(c)\frac{dk}{dr} + wu''(c)\frac{dc}{dr}\right], \\
&= \frac{1}{v''(l)}\left[-ku'(c) + wu''(c)\frac{dc}{dr}\right].
\end{aligned}$$

Finally, from the optimality condition for consumption (8) with $\dot{c}_t = 0$ and $a_t - (b_t + m_t) = K = kl$, we have:

$$\begin{aligned}
\frac{dc}{dr} &= \frac{1}{u''(c)}\left[\frac{\gamma'(kl)}{(\rho - r)^2} + \frac{\gamma''(kl)}{\rho - r}\left[l\frac{dk}{dr} + k\frac{dl}{dr}\right]\right], \\
&= \frac{u'(c)}{u''(c)}\left[\frac{u'(c)}{\gamma'(kl)} + \frac{\gamma''(kl)}{\gamma'(kl)}\left[\frac{l}{f''(k)} + \frac{k}{v''(l)}\left[-ku'(c) + wu''(c)\frac{dc}{dr}\right]\right]\right], \\
&= \frac{u'(c)}{u''(c)}\left[\frac{u'(c)}{\gamma'(kl)} - \frac{kl\gamma''(kl)}{\gamma'(kl)}\left(\frac{ku'(c)}{lv''(l)} - \frac{1}{kf''(k)}\right)\right] + \frac{kl\gamma''(kl)}{\gamma'(kl)}\frac{v'(l)}{lv''(l)}\frac{dc}{dr}.
\end{aligned}$$

where I have used the fact that $\rho - r = \gamma'(kl)/u'(c)$ to derive the second line and that $wu'(c) = v'(l)$ to derive the third line. Rearranging terms, this expression can be written as:

$$\frac{dc}{dr} = \frac{-1}{\frac{lv''(l)}{v'(l)} + \frac{-kl\gamma''(kl)}{\gamma'(kl)}}\frac{lv''(l)}{v'(l)}\frac{u'(c)}{-u''(c)}\left[\frac{u'(c)}{\gamma'(kl)} + \frac{-kl\gamma''(kl)}{\gamma'(kl)}\left(\frac{ku'(c)}{lv''(l)} + \frac{1}{-kf''(k)}\right)\right],$$

which shows that consumption is unambiguously decreasing in the real interest rate.

Substituting this expression into the derivative of labor supply yields:

$$\begin{aligned}
\frac{dl}{dr} &= \frac{-ku'(c)}{v''(l)} + \frac{l}{\frac{lv''(l)}{v'(l)} + \frac{-kl\gamma''(kl)}{\gamma'(kl)}} \left[\frac{u'(c)}{\gamma'(kl)} + \frac{-kl\gamma''(kl)}{\gamma'(kl)} \left(\frac{ku'(c)}{lv''(l)} + \frac{1}{-kf''(k)} \right) \right], \\
&= \frac{l}{\frac{lv''(l)}{v'(l)} + \frac{-kl\gamma''(kl)}{\gamma'(kl)}} \left[\frac{u'(c)}{\gamma'(kl)} - \frac{lv''(l)}{v'(l)} \frac{ku'(c)}{lv''(l)} + \frac{-kl\gamma''(kl)}{\gamma'(kl)} \frac{1}{-kf''(k)} \right], \\
&= \frac{1}{\frac{lv''(l)}{v'(l)} + \frac{-kl\gamma''(kl)}{\gamma'(kl)}} \left[\frac{lu'(c)}{\gamma'(kl)} \left(1 - \frac{k\gamma'(kl)}{v'(l)} \right) + \frac{-kl\gamma''(kl)}{\gamma'(kl)} \frac{l}{-kf''(k)} \right].
\end{aligned}$$

But, as $\gamma'(kl) = (\rho - r)\, u'(c) = (\rho - f'(k) + \delta)\, u'(c)$ and $v'(l) = wu'(c)$, we have:

$$\begin{aligned}
\frac{dl}{dr} &= \frac{1}{\frac{lv''(l)}{v'(l)} + \frac{-kl\gamma''(kl)}{\gamma'(kl)}} \left[\frac{lu'(c)}{\gamma'(kl)} \left(1 - \frac{k(\rho - f'(k) + \delta)}{w} \right) + \frac{-kl\gamma''(kl)}{\gamma'(kl)} \frac{l}{-kf''(k)} \right], \\
&= \frac{1}{\frac{lv''(l)}{v'(l)} + \frac{-kl\gamma''(kl)}{\gamma'(kl)}} \left[\frac{lu'(c)}{\gamma'(kl)} \frac{f(k) - k(\rho + \delta)}{w} + \frac{-kl\gamma''(kl)}{\gamma'(kl)} \frac{l}{-kf''(k)} \right],
\end{aligned}$$

where I have used the fact that $w = f(k) - kf'(k)$. Clearly, if labor supply is sufficiently inelastic, then dl/dr can be arbitrarily close to zero.

Let y^s and y^d denote the aggregate supply and the aggregate demand, respectively. We have $y^s = F(K, L) = F(k, 1)\, L = f(k)\, l$, since $L = l$, and $y^d = c + \delta K = c + \delta kl$. We can now compare the slopes of the aggregate demand and supply curves:

$$\begin{aligned}
\frac{d\left(y^s - y^d\right)}{dr} &= \frac{d\left([f(k) - \delta k]\, l - c\right)}{dr}, \\
&= [f(k) - \delta k] \frac{dl}{dr} + [f'(k) - \delta]\, l \frac{dk}{dr} - \frac{dc}{dr}, \\
&= [f(k) - \delta k] \frac{dl}{dr} + \frac{rl}{f''(k)} - \frac{dc}{dr}. \qquad \text{(B1)}
\end{aligned}$$

where, to get the third line, I have used the fact that $f'(k) - \delta = r$. If labor supply is inelastic, i.e. $dl/dr = 0$, we have:

$$\frac{d\left(y^s - y^d\right)}{dr} = \frac{rl}{f''(k)} - \frac{dc}{dr} \geq \frac{\rho l}{f''(k)} - \frac{dc}{dr},$$

where I have used the fact that $\rho - r = \gamma'(kl)/u'(c) \geq 0$. Hence, if $\rho = 0$:

$$\frac{d\left(y^s - y^d\right)}{dr} \geq -\frac{dc}{dr} > 0,$$

or, equivalently:

$$\frac{dy^s}{dr} > \frac{dy^d}{dr}.$$

But, recall that we have $r = i - \pi > 0 - \bar{\pi} = r^n$ and that $y^s = y^d$ when $r = r^n$. The inequality therefore implies that y^d is strictly smaller than y^s for all $r > r^n$. By continuity, this must be true provided that ρ is sufficiently close to zero and that labor supply is sufficiently inelastic.

Even if labor supply is very elastic, the first term of (B1) is positive under mild conditions, which strengthens the above result. More formally, we have:

$$[f(k) - \delta k] \frac{dl}{dr} = \frac{y^s - y^d + c}{l} \frac{1}{\frac{lv''(l)}{v'(l)} - \frac{kl\gamma''(kl)}{\gamma'(kl)}} \left[\frac{y^s - y^d + c - \rho kl}{(\rho + \pi) w} + \frac{l}{kf''(k)} \frac{kl\gamma''(kl)}{\gamma'(kl)} \right].$$

where I have used the fact that $y^s - y^d = [f(k) - \delta k] l - c$. If $c > \rho K$ for all $r \in [r^n, \rho]$, then this expression is positive for $r = r^n$, since $y^s = y^d$ when $r = r^n$. This implies that $dy^s/dr > dy^d/dr$ for $r = r^n$. By induction, as r increases from r^n to ρ, we have $y^d < y^s$, $[f(k) - \delta k] dl/dr > 0$, and $dy^s/dr > dy^d/dr$ for all $r \in [r^n, \rho]$.

C Proof of Lemma 3

As in the proof of the previous two lemmas, let $k = K/L$ and $f(k) = F(k, 1)$. By (8), (9), $-\pi = f'(k) - \delta$, $w = f(k) - kf'(k)$, and (36), we know that c, l, k, w, and L are continuous functions of $r = -\pi$ for $\pi \in \left(\pi^R - \alpha, \bar{\pi}\right]$.[46] Hence, aggregate demand $y^d = c + \delta kL$ and aggregate supply $y^s = Lf(k)$ are continuous functions of π for $\pi \in \left(\pi^R - \alpha, \bar{\pi}\right]$. By assumption, $y^d < y^s$ for $\pi = \pi^R$. Thus, to prove the existence of an equilibrium, it is sufficient to show that, in the limit as π tends to $\pi^R - \alpha$, we have $y^d > y^s$.

Let $\tilde{f}(k) = f(k) - kf'(k)$. We know that, for any K and L, $F_L(K, L) = f(K/L) - K/Lf'(K/L) = \tilde{f}(K/L)$. We also have $F(0, L) = 0$ for any L, which implies that $F_L(0, L) = 0$. Thus, $F_L(0, L) = \tilde{f}(0/L) = \tilde{f}(0) = 0$. We also have $\tilde{f}'(k) = -kf''(k)$, which shows that $\tilde{f}(k)$ is strictly increasing for any $k > 0$. Hence, the unique solution to $\tilde{f}(k) = 0$ for $k \geq 0$ is $k = 0$.

To ensure that the demand for consumption, given by (8), is strictly positive when $\pi = \pi^R - \alpha$, we need to have $\rho + \pi = \rho + \pi^R - \alpha > 0$ or, equivalently, $\alpha < \pi^R + \rho$. This in fact guarantees that consumption is positive for all $\pi \in \left(\pi^R - \alpha, \pi^R\right)$. Thus, by (8), $-\pi = f'(k) - \delta$, $w = f(k) - kf'(k)$, and (9), the values of c, k, w, and l are strictly positive and finite when $\pi = \pi^R - \alpha$. The remaining variable, L, is implicitly determined

[46] Recall that, by footnote 12, $\bar{\pi} < \delta$. This guarantees that $-\pi = f'(k) - \delta$ defines k as a continuous function of π for all $\pi \in \left(\pi^R - \alpha, \bar{\pi}\right]$. Also, note that the consumer's problem implies that we have a corner solution with $c = 0$ and $v'(l) = +\infty$ whenever $\pi \leq -\rho$. However, this case does not arise when $\alpha < \pi^R + \rho$ since $\pi > \pi^R - \alpha > -\rho$.

by (36), which can be written as:

$$\begin{aligned} w &= \frac{\alpha}{\pi - (\pi^R - \alpha)} F_L(K, l), \\ &= \frac{\alpha}{\pi - (\pi^R - \alpha)} \tilde{f}\left(\frac{K}{l}\right), \\ &= \frac{\alpha}{\pi - (\pi^R - \alpha)} \tilde{f}\left(\frac{L}{l} k\right). \end{aligned}$$

But, as π tends to $\pi^R - \alpha$, the first term of the product tends to infinity while w tends to a strictly positive and finite value. Hence, $\tilde{f}(kL/l)$ must tend to zero while k and l also tend to strictly positive and finite values. Thus, labor demand L must tend to zero as π tends to $\pi^R - \alpha$. But, if L tends to zero while k and c tend to strictly positive and finite values, then $y^s = Lf(k)$ must tend to zero while $y^d = c + \delta k L$ tends to $c > 0$. Thus, in the limit, we unambiguously have $y^d > y^s$.

As α tends to zero, $\pi^R - \alpha$ tends to π^R. Thus, c, k, w, and l for $\pi \in (\pi^R - \alpha, \pi^R)$ converge to strictly positive and finite values (since α does not show up in (8), $-\pi = f'(k) - \delta$, $w = f(k) - kf'(k)$, or (9) while π converges to π^R). There is therefore a unique value of L that can equate aggregate supply $y^s = Lf(k)$ to demand $y^d = c + \delta k L$. But, for any $\alpha > 0$ and $\pi \in (\pi^R - \alpha, \pi^R)$, L is implicitly given by:

$$w = \frac{\alpha}{\pi - (\pi^R - \alpha)} \tilde{f}\left(\frac{L}{l} k\right).$$

Thus, the unique value L that equates y^s and y^d determines a unique value of the ratio $\alpha / [\pi - (\pi^R - \alpha)]$. Hence, for any α strictly positive but arbitrarily small, there is a unique corresponding rate of inflation $\pi \in (\pi^R - \alpha, \pi^R)$ that is consistent with an equilibrium.

D Lemma 4

As in the proof of Lemma 3, let $k = K/L$, $f(k) = F(k, 1)$, and $\tilde{f}(k) = f(k) - kf'(k)$. In a secular stagnation steady state with an exogenous real interest rate r, we must have $u'(c) = \gamma'(kL) / (\rho - r)$, by (8), $r = f'(k) - \delta$, by (16), and $L[f(k) - \delta k] = c$, by (24).[47]

[47] The real interest rate does not directly affect the other equations that characterize the secular stagnation steady state.

Totally differentiating these last two equations with respect to r yields:

$$\begin{aligned}\frac{dk}{dr} &= \frac{1}{f''(k)},\\ \frac{dL}{dr} &= \frac{1}{f(k)-\delta k}\frac{dc}{dr} - L\frac{f'(k)-\delta}{f(k)-\delta k}\frac{dk}{dr},\end{aligned}$$

respectively. It follows that:

$$\begin{aligned}k\frac{dL}{dr} + L\frac{dk}{dr} &= \frac{k}{f(k)-\delta k}\frac{dc}{dr} - Lk\frac{f'(k)-\delta}{f(k)-\delta k}\frac{dk}{dr} + L\frac{dk}{dr},\\ &= \frac{k}{f(k)-\delta k}\frac{dc}{dr} + L\frac{f(k)-kf'(k)}{f(k)-\delta k}\frac{dk}{dr},\\ &= \frac{kL}{c}\frac{dc}{dr} + \frac{L^2}{c}\frac{\tilde{f}(k)}{f''(k)},\end{aligned}$$

where I have used the fact that $f(k)-\delta k = c/L$. Totally differentiating $u'(c) = \gamma'(kL)/(\rho-r)$ yields:

$$\begin{aligned}\frac{dc}{dr} &= \frac{1}{u''(c)}\left[\frac{\gamma'(kL)}{(\rho-r)^2} + \frac{\gamma''(kL)}{\rho-r}\left[k\frac{dL}{dr} + L\frac{dk}{dr}\right]\right],\\ &= \frac{\gamma'(kL)}{u''(c)(\rho-r)}\left[\frac{u'(c)}{\gamma'(kL)} + \frac{\gamma''(kL)}{\gamma'(kL)}\left[\frac{kL}{c}\frac{dc}{dr} + \frac{L^2}{c}\frac{\tilde{f}(k)}{f''(k)}\right]\right],\\ &= \frac{u'(c)}{cu''(c)}\left[\frac{cu'(c)}{\gamma'(kL)} + \frac{kL\gamma''(kL)}{\gamma'(kL)}\frac{L\tilde{f}(k)}{kf''(k)}\right] + \frac{u'(c)}{cu''(c)}\frac{kL\gamma''(kL)}{\gamma'(kL)}\frac{dc}{dr},\end{aligned}$$

where I have used the fact that $\gamma'(kL)/(\rho-r) = u'(c)$. It follows that:

$$\frac{dc}{dr} = \frac{1}{\varepsilon_\gamma - \varepsilon_u}\left[\frac{cu'(c)}{\gamma'(kL)} - \varepsilon_\gamma\frac{L\tilde{f}(k)}{kf''(k)}\right],$$

where $\varepsilon_\gamma = -kL\gamma''(kL)/\gamma'(kL)$ and $\varepsilon_u = -cu''(c)/u'(c)$. Thus, $dc/dr < 0$ if and only if $\varepsilon_\gamma < \varepsilon_u$.

We therefore have:

$$\begin{aligned}k\frac{dL}{d\alpha} + L\frac{dk}{d\alpha} &= \frac{kL}{c}\left[\frac{dc}{dr} + \frac{L}{k}\frac{\tilde{f}(k)}{f''(k)}\right],\\ &= \frac{kL}{c}\frac{1}{\varepsilon_\gamma - \varepsilon_u}\left[\frac{cu'(c)}{\gamma'(kL)} - \varepsilon_\gamma\frac{L\tilde{f}(k)}{kf''(k)} + (\varepsilon_\gamma - \varepsilon_u)\frac{L}{k}\frac{\tilde{f}(k)}{f''(k)}\right],\\ &= \frac{kL}{c}\frac{1}{\varepsilon_\gamma - \varepsilon_u}\left[\frac{cu'(c)}{\gamma'(kL)} - \varepsilon_u\frac{L}{k}\frac{\tilde{f}(k)}{f''(k)}\right].\end{aligned}$$

Output is given by $y = c + \delta kL$, which implies that:

$$\begin{aligned}
\frac{dy}{dr} &= \frac{dc}{dr} + \delta\left[k\frac{dL}{dr} + L\frac{dk}{dr}\right], \\
&= \frac{1}{\varepsilon_\gamma - \varepsilon_u}\left[\frac{cu'(c)}{\gamma'(kL)} - \varepsilon_\gamma \frac{L\tilde{f}(k)}{kf''(k)} + \frac{\delta kL}{c}\left[\frac{cu'(c)}{\gamma'(kL)} - \varepsilon_u \frac{L}{k}\frac{\tilde{f}(k)}{f''(k)}\right]\right], \\
&= \frac{1}{\varepsilon_u - \varepsilon_\gamma}\left[\frac{c+\delta kL}{c}\frac{cu'(c)}{\gamma'(kL)} - \left(\varepsilon_\gamma + \frac{\delta kL}{c}\varepsilon_u\right)\frac{L\tilde{f}(k)}{kf''(k)}\right].
\end{aligned}$$

Hence, $dy/dr < 0$ if and only if $\varepsilon_\gamma < \varepsilon_u$.

E Proofs of Proposition 1, 2, 3, 4 and 5

As in the proof of Lemma 3, let $k = K/L$, $f(k) = F(k,1)$, and $\tilde{f}(k) = f(k) - kf'(k)$. We know that $\tilde{f}(0) = 0$, that $\tilde{f}'(k) = -kf''(k) > 0$ for any $k > 0$, and that, for any K and L, $F_L(K,L) = f(K/L) - K/Lf'(K/L) = \tilde{f}(K/L)$. Combining (15) and (36), we therefore have:

$$\begin{aligned}
F_L(K,L) &= \frac{\alpha}{\pi - (\pi^R - \alpha)} F_L(K,l), \\
\tilde{f}\left(\frac{K}{L}\right) &= \frac{\alpha}{\pi - (\pi^R - \alpha)} \tilde{f}\left(\frac{K}{l}\right), \\
\left[\pi - \left(\pi^R - \alpha\right)\right]\tilde{f}(k) &= \alpha \tilde{f}\left(\frac{L}{l}k\right).
\end{aligned}$$

We also know, by (15), that $w = \tilde{f}(k)$. The secular stagnation equilibrium (c,k,π,l,L,y) is therefore fully characterized by:

$$u'(c) = \frac{\chi\gamma'(kL)}{\rho + \pi}, \tag{E1}$$

$$-\pi = \xi f'(k) - \delta, \tag{E2}$$

$$\left[\pi - \left(\pi^R - \alpha\right)\right]\tilde{f}(k) = \alpha\tilde{f}\left(\frac{L}{l}k\right), \tag{E3}$$

$$\varrho v'(l) = \xi\tilde{f}(k)u'(c), \tag{E4}$$

$$\xi Lf(k) = c + \delta kL + g, \tag{E5}$$

$$y = c + \delta kL + g, \tag{E6}$$

where I have added three new parameters, χ, ϱ, and ξ, that stand for the marginal utility of wealth, for the disutility of supplying labor, and for total factor productivity, respectively. To prove the propositions, we need to totally differentiate these equations with respect to α, χ, ϱ, ξ, and g. I set $\chi = 1$, except in the proof of Proposition 2, $\varrho = \xi = 1$, except in the proof of Proposition 3, and $g = 0$, except in the proof of the Proposition 4 and 5.

Finally, throughout the proofs, I will use the notations $\varepsilon_u = -cu''(c)/u'(c)$ and $\varepsilon_\gamma = -K\gamma''(K)/\gamma'(K)$.

E.1 Proof of Proposition 1

Totally differentiating (E2) and (E5) with respect to α yields:

$$\begin{aligned}
\frac{dk}{d\alpha} &= \frac{1}{-f''(k)}\frac{d\pi}{d\alpha}, \\
\frac{dL}{d\alpha} &= \frac{1}{f(k)-\delta k}\frac{dc}{d\alpha} - L\frac{f'(k)-\delta}{f(k)-\delta k}\frac{dk}{d\alpha},
\end{aligned}$$

respectively. It follows that:

$$\begin{aligned}
k\frac{dL}{d\alpha} + L\frac{dk}{d\alpha} &= \frac{k}{f(k)-\delta k}\frac{dc}{d\alpha} - Lk\frac{f'(k)-\delta}{f(k)-\delta k}\frac{dk}{d\alpha} + L\frac{dk}{d\alpha}, \\
&= \frac{k}{f(k)-\delta k}\frac{dc}{d\alpha} + L\frac{f(k)-kf'(k)}{f(k)-\delta k}\frac{dk}{d\alpha}, \\
&= \frac{kL}{c}\frac{dc}{d\alpha} - \frac{L^2}{c}\frac{\tilde{f}(k)}{f''(k)}\frac{d\pi}{d\alpha},
\end{aligned}$$

where I have used the fact that $f(k) - \delta k = c/L$. Totally differentiating (E1) yields:

$$\begin{aligned}
\frac{dc}{d\alpha} &= \frac{1}{u''(c)}\left[\frac{-\gamma'(kL)}{(\rho+\pi)^2}\frac{d\pi}{d\alpha} + \frac{\gamma''(kL)}{\rho+\pi}\left[k\frac{dL}{d\alpha} + L\frac{dk}{d\alpha}\right]\right], \\
&= \frac{\gamma'(kL)}{u''(c)(\rho+\pi)}\left[\frac{-u'(c)}{\gamma'(kL)}\frac{d\pi}{d\alpha} + \frac{\gamma''(kL)}{\gamma'(kL)}\left[\frac{kL}{c}\frac{dc}{d\alpha} - \frac{L^2}{c}\frac{\tilde{f}(k)}{f''(k)}\frac{d\pi}{d\alpha}\right]\right], \\
&= \frac{-u'(c)}{cu''(c)}\left[\frac{cu'(c)}{\gamma'(kL)} + \frac{kL\gamma''(kL)}{\gamma'(kL)}\frac{L\tilde{f}(k)}{kf''(k)}\right]\frac{d\pi}{d\alpha} + \frac{u'(c)}{cu''(c)}\frac{kL\gamma''(kL)}{\gamma'(kL)}\frac{dc}{d\alpha},
\end{aligned}$$

where I have used the fact that $\gamma'(kL)/(\rho+\pi) = u'(c)$. It follows that:

$$\frac{dc}{d\alpha} = \frac{1}{\varepsilon_u - \varepsilon_\gamma}\left[\frac{cu'(c)}{\gamma'(kL)} - \varepsilon_\gamma\frac{L\tilde{f}(k)}{kf''(k)}\right]\frac{d\pi}{d\alpha},$$

where $\varepsilon_u = -cu''(c)/u'(c)$ and $\varepsilon_\gamma = -kL\gamma''(kL)/\gamma'(kL)$. Hence:

$$
\begin{aligned}
k\frac{dL}{d\alpha} + L\frac{dk}{d\alpha} &= \frac{kL}{c}\left[\frac{dc}{d\alpha} - \frac{L}{k}\frac{\tilde{f}(k)}{f''(k)}\frac{d\pi}{d\alpha}\right], \\
&= \frac{kL}{c}\frac{1}{\varepsilon_u - \varepsilon_\gamma}\left[\frac{cu'(c)}{\gamma'(kL)} - \varepsilon_\gamma \frac{L\tilde{f}(k)}{kf''(k)} - (\varepsilon_u - \varepsilon_\gamma)\frac{L}{k}\frac{\tilde{f}(k)}{f''(k)}\right]\frac{d\pi}{d\alpha}, \\
&= \frac{kL}{c}\frac{1}{\varepsilon_u - \varepsilon_\gamma}\left[\frac{cu'(c)}{\gamma'(kL)} - \varepsilon_u \frac{L}{k}\frac{\tilde{f}(k)}{f''(k)}\right]\frac{d\pi}{d\alpha}.
\end{aligned}
$$

Differentiating (E6) with respect to α gives:

$$
\begin{aligned}
\frac{dy}{d\alpha} &= \frac{dc}{d\alpha} + \delta\left[k\frac{dL}{d\alpha} + L\frac{dk}{d\alpha}\right], \\
&= \frac{1}{\varepsilon_u - \varepsilon_\gamma}\left[\frac{cu'(c)}{\gamma'(kL)} - \varepsilon_\gamma\frac{L\tilde{f}(k)}{kf''(k)} + \frac{\delta kL}{c}\left[\frac{cu'(c)}{\gamma'(kL)} - \varepsilon_u\frac{L}{k}\frac{\tilde{f}(k)}{f''(k)}\right]\right]\frac{d\pi}{d\alpha}, \\
&= \frac{1}{\varepsilon_u - \varepsilon_\gamma}\left[\frac{c + \delta kL}{c}\frac{cu'(c)}{\gamma'(kL)} - \left(\varepsilon_\gamma + \frac{\delta kL}{c}\varepsilon_u\right)\frac{L\tilde{f}(k)}{kf''(k)}\right]\frac{d\pi}{d\alpha}.
\end{aligned}
$$

If $\varepsilon_u - \varepsilon_\gamma > 0$, then the term in the main square bracket is strictly positive. In that case, a fall in inflation results in a lower output level.

Totally differentiating (E3) with respect to α yields:

$$
\tilde{f}(k) + \tilde{f}(k)\frac{d\pi}{d\alpha} + \left[\pi - \left(\pi^R - \alpha\right)\right]\tilde{f}'(k)\frac{dk}{d\alpha} = \tilde{f}\left(\frac{L}{l}k\right) + \alpha\tilde{f}'\left(\frac{L}{l}k\right)\left[\frac{L}{l}\frac{dk}{d\alpha} + \frac{k}{l}\frac{dL}{d\alpha} - \frac{Lk}{l^2}\frac{dl}{d\alpha}\right],
$$

where $dl/d\alpha$ can be computed from (E4). We know that, in the secular stagnation equilibrium, $\pi \in \left(\pi^R - \alpha, \pi^R\right)$. Hence, as α tends to zero, $\pi - \left(\pi^R - \alpha\right)$ also tends to zero. It follows that, as α tends to zero, the above expression simplifies to:

$$
\frac{d\pi}{d\alpha} = \frac{\tilde{f}\left(\frac{L}{l}k\right) - \tilde{f}(k)}{\tilde{f}(k)}.
$$

But, $L < l$ and $\tilde{f}(\cdot)$ is a strictly increasing function.[48] Hence, $d\pi/d\alpha < 0$. By continuity, this result remains true if α is sufficiently close to zero.

[48] In the limit as α tends to zero, we know that $L < l$. Indeed, when $\pi = \pi^R$, we know by Lemma 2 that $y^s = lf(k) > c + \delta kl = y^d$. If $\pi \in \left(\pi^R - \alpha, \pi^R\right)$, then, in the limit as α tends to zero, c, l, or k are not affected by the fact that $\pi < \pi^R$ (and we still have $lf(k) > c + \delta kl$), while in the secular stagnation equilibrium L is such that $Lf(k) = c + \delta kL$. Combining these two equations immediately yields $l > L$.

E.2 Proof of Proposition 2

Differentiating (E3) with respect to χ gives:

$$\tilde{f}(k)\frac{d\pi}{d\chi}+\left[\pi-\left(\pi^R-\alpha\right)\right]\tilde{f}'(k)\frac{dk}{d\chi}=\frac{\alpha}{l}\tilde{f}'\left(\frac{L}{l}k\right)\left[L\frac{dk}{d\chi}+k\frac{dL}{d\chi}-\frac{Lk}{l}\frac{dl}{d\chi}\right].$$

But, given that in the secular stagnation equilibrium $\pi\in\left(\pi^R-\alpha,\pi^R\right)$, we know that $\pi-\left(\pi^R-\alpha\right)$ tends to zero as α tends to zero. Also, $dc/d\chi$, $dk/d\chi$, $dl/d\chi$, and $dL/d\chi$, which can be derived from (E1), (E2), (E4), and (E5), respectively, are only affected by α through $d\pi/d\chi$. Hence, as α tends to zero, $d\pi/d\chi$ also tends to zero.

Relying on (E2) and (E5), where χ does not appear, and proceeding as in the proof of Proposition 1, we obtain:

$$k\frac{dL}{d\chi}+L\frac{dk}{d\chi}=\frac{kL}{c}\frac{dc}{d\chi}-\frac{L^2}{c}\frac{\tilde{f}(k)}{f''(k)}\frac{d\pi}{d\chi}.$$

Differentiating (E1) with respect to χ gives:

$$\begin{aligned}\frac{dc}{d\chi}&=\frac{1}{u''(c)}\left[\frac{\gamma'(kL)}{\rho+\pi}-\frac{\chi\gamma'(kL)}{(\rho+\pi)^2}\frac{d\pi}{d\chi}+\frac{\chi\gamma''(kL)}{\rho+\pi}\left[k\frac{dL}{d\chi}+L\frac{dk}{d\chi}\right]\right],\\&=\frac{1}{cu''(c)}\frac{\chi\gamma'(kL)}{\rho+\pi}\left[\frac{c}{\chi}+\left(\varepsilon_\gamma\frac{L\tilde{f}(k)}{kf''(k)}-\frac{c}{\rho+\pi}\right)\frac{d\pi}{d\chi}-\varepsilon_\gamma\frac{dc}{d\chi}\right],\\&=\frac{-1}{\varepsilon_u}\left[\frac{c}{\chi}+\left(\varepsilon_\gamma\frac{L\tilde{f}(k)}{kf''(k)}-\frac{cu'(c)}{\chi\gamma'(kL)}\right)\frac{d\pi}{d\chi}\right]+\frac{\varepsilon_\gamma}{\varepsilon_u}\frac{dc}{d\chi}.\end{aligned}$$

where, as before, $\varepsilon_u=-cu''(c)/u'(c)$ and $\varepsilon_\gamma=-kL\gamma''(kL)/\gamma'(kL)$. It follows that:

$$\frac{dc}{d\chi}=\frac{-1}{\varepsilon_u-\varepsilon_\gamma}\left[\frac{c}{\chi}+\left(\varepsilon_\gamma\frac{L\tilde{f}(k)}{kf''(k)}-\frac{cu'(c)}{\chi\gamma'(kL)}\right)\frac{d\pi}{d\chi}\right].$$

Hence:

$$\begin{aligned}\frac{d(\delta kL)}{d\chi}&=\delta\frac{kL}{c}\frac{dc}{d\chi}-\delta\frac{L^2}{c}\frac{\tilde{f}(k)}{f''(k)}\frac{d\pi}{d\chi},\\&=\frac{-1}{\varepsilon_u-\varepsilon_\gamma}\left[\frac{\delta kL}{\chi}+\left(\varepsilon_\gamma\frac{\delta L^2}{c}\frac{\tilde{f}(k)}{f''(k)}-\frac{\delta kLu'(c)}{\chi\gamma'(kL)}+(\varepsilon_u-\varepsilon_\gamma)\frac{\delta L^2}{c}\frac{\tilde{f}(k)}{f''(k)}\right)\frac{d\pi}{d\chi}\right],\\&=\frac{-1}{\varepsilon_u-\varepsilon_\gamma}\left[\frac{\delta kL}{\chi}+\left(\varepsilon_u\frac{\delta L^2}{c}\frac{\tilde{f}(k)}{f''(k)}-\frac{\delta kLu'(c)}{\chi\gamma'(kL)}\right)\frac{d\pi}{d\chi}\right].\end{aligned}$$

Recall that $d\pi/d\chi$ tends to zero as α tends to zero.[49] It unambiguously follows that

[49] Using the above expressions, it is possible to show that for α strictly positive, but sufficiently small,

$dc/d\chi < 0$ and $d\left(\delta kL\right)/d\chi < 0$, provided that $\varepsilon_u > \varepsilon_\gamma$.

E.3 Proof of Proposition 3

I first focus on the impact of the disutility of labor ϱ while setting $\xi = 1$; I then focus on the impact of total factor productivity ξ while setting $\varrho = 1$.

E.3.1 Disutility of Labor

Relying on (E1), (E2), and (E5), where ϱ does not appear, and proceeding as in the proof of Proposition 1, we obtain:

$$\frac{dk}{d\delta} = \frac{1}{-f''\left(k\right)}\frac{d\pi}{d\varrho},$$

$$\frac{dc}{d\varrho} = \frac{1}{\varepsilon_u - \varepsilon_\gamma}\left[\frac{cu'\left(c\right)}{\gamma'\left(kL\right)} - \varepsilon_\gamma\frac{L\tilde{f}\left(k\right)}{kf''\left(k\right)}\right]\frac{d\pi}{d\varrho},$$

$$k\frac{dL}{d\varrho} + L\frac{dk}{d\varrho} = \frac{kL}{c}\frac{1}{\varepsilon_u - \varepsilon_\gamma}\left[\frac{cu'\left(c\right)}{\gamma'\left(kL\right)} - \varepsilon_u\frac{L}{k}\frac{\tilde{f}\left(k\right)}{f''\left(k\right)}\right]\frac{d\pi}{d\varrho}.$$

If $\varepsilon_u > \varepsilon_\gamma$ and if a rise in the disutility of labor is inflationary, then, by the second expression, it must increase consumption c while, by the third expression, it must raise investment δkL. Totally differentiating (E4) with respect to ϱ yields:

$$\begin{aligned}\frac{dl}{d\varrho} &= \frac{1}{\varrho v''\left(l\right)}\left[-v'\left(l\right) - kf''\left(k\right)u'\left(c\right)\frac{dk}{d\varrho} + \tilde{f}\left(k\right)u''\left(c\right)\frac{dc}{d\varrho}\right],\\ &= \frac{-v'\left(l\right)}{\varrho v''\left(l\right)} + \frac{1}{\varrho v''\left(l\right)}\left[ku'\left(c\right) + \frac{\tilde{f}\left(k\right)u''\left(c\right)}{\varepsilon_u - \varepsilon_\gamma}\left(\frac{cu'\left(c\right)}{\gamma'\left(kL\right)} - \varepsilon_\gamma\frac{L\tilde{f}\left(k\right)}{kf''\left(k\right)}\right)\right]\frac{d\pi}{d\varrho}.\end{aligned}$$

Differentiating (E3), we have:

$$\tilde{f}\left(k\right)\frac{d\pi}{d\varrho} - \left[\pi - \left(\pi^R - \alpha\right)\right]kf''\left(k\right)\frac{dk}{d\varrho} = \frac{\alpha}{l}\tilde{f}'\left(\frac{L}{l}k\right)\left[L\frac{dk}{d\varrho} + k\frac{dL}{d\varrho} - \frac{kL}{l}\frac{dl}{d\varrho}\right].$$

Substituting the above expressions into this equation gives:

$$\tilde{f}\left(k\right)\frac{d\pi}{d\varrho} + \left[\pi - \left(\pi^R - \alpha\right)\right]k\frac{d\pi}{d\varrho} = \alpha\tilde{f}'\left(\frac{L}{l}k\right)\frac{kL}{l^2}\frac{v'\left(l\right)}{\varrho v''\left(l\right)} + \frac{\alpha}{l}\tilde{f}'\left(\frac{L}{l}k\right)\left[...\right]\frac{d\pi}{d\varrho}.$$

As α tends to zero, $\pi - \left(\pi^R - \alpha\right)$ also tends to zero. Thus, solving for $d\pi/d\varrho$ shows that it tends to:

$$\frac{1}{\tilde{f}\left(k\right)}\alpha\tilde{f}'\left(\frac{L}{l}k\right)\frac{kL}{l^2}\frac{v'\left(l\right)}{\varrho v''\left(l\right)},$$

$d\pi/d\gamma < 0$, which amplifies the fall in consumption and in investment.

which is strictly positive provided that $\alpha > 0$.[50] Thus, for α sufficiently small but strictly positive, we have $d\pi/d\varrho > 0$. It follows that a rise in the disutility of labor increases inflation, consumption, investment, and, hence, output.

E.3.2 Total Factor Productivity

Totally differentiating (E2) and (E5) with respect to ξ yields:

$$\begin{aligned}
\frac{dk}{d\xi} &= \frac{-f'(k)}{\xi f''(k)} - \frac{1}{\xi f''(k)}\frac{d\pi}{d\xi}, \\
\frac{dL}{d\xi} &= \frac{1}{\xi f(k) - \delta k}\left[-Lf(k) + \frac{dc}{d\xi} - L\left[\xi f'(k) - \delta\right]\frac{dk}{d\xi}\right], \\
&= \frac{-L^2 f(k)}{c} + \frac{L^2}{c}\frac{f'(k)\left[\xi f'(k) - \delta\right]}{\xi f''(k)} + \frac{L}{c}\frac{dc}{d\xi} + \frac{L^2}{c}\frac{\left[\xi f'(k) - \delta\right]}{\xi f''(k)}\frac{d\pi}{d\xi}.
\end{aligned}$$

It follows that:

$$\begin{aligned}
k\frac{dL}{d\xi} + L\frac{dk}{d\xi} &= \frac{-kL^2}{c}f(k) + \frac{kL^2}{c}\frac{f'(k)\left[\xi f'(k) - \delta\right]}{\xi f''(k)} - \frac{Lf'(k)}{\xi f''(k)} + \frac{kL}{c}\frac{dc}{d\xi} \\
&\quad + \left[\frac{kL^2}{c}\frac{\left[\xi f'(k) - \delta\right]}{\xi f''(k)} - \frac{L}{\xi f''(k)}\right]\frac{d\pi}{d\xi}, \\
&= \frac{-kL^2}{c}f(k) + \frac{L^2}{c}\frac{f'(k)}{\xi f''(k)}\left[k\left[\xi f'(k) - \delta\right] - \left[\xi f(k) - \delta k\right]\right] + \frac{kL}{c}\frac{dc}{d\xi} \\
&\quad + \frac{L^2}{c}\frac{1}{\xi f''(k)}\left[k\left[\xi f'(k) - \delta\right] - \left[\xi f(k) - \delta k\right]\right]\frac{d\pi}{d\xi}, \\
&= \frac{-L^2}{c}\left[kf(k) + \frac{f'(k)}{f''(k)}\tilde{f}(k)\right] + \frac{kL}{c}\frac{dc}{d\xi} - \frac{L^2}{c}\frac{\tilde{f}(k)}{f''(k)}\frac{d\pi}{d\xi}.
\end{aligned}$$

With a Cobb-Douglas aggregate production function, we have $f(k) = k^{\varsigma}$ for some $\varsigma \in [0,1]$. This implies that $kf(k)f''(k) + f'(k)\tilde{f}(k) = 0$. Hence, proceeding as in the proof of Proposition 1, we obtain:

$$\begin{aligned}
\frac{dc}{d\xi} &= \frac{1}{\varepsilon_u - \varepsilon_\gamma}\left[\frac{cu'(c)}{\gamma'(kL)} - \varepsilon_\gamma\frac{L\tilde{f}(k)}{kf''(k)}\right]\frac{d\pi}{d\xi}, \\
k\frac{dL}{d\xi} + L\frac{dk}{d\xi} &= \frac{kL}{c}\frac{1}{\varepsilon_u - \varepsilon_\gamma}\left[\frac{cu'(c)}{\gamma'(kL)} - \varepsilon_u\frac{L}{k}\frac{\tilde{f}(k)}{f''(k)}\right]\frac{d\pi}{d\xi}.
\end{aligned}$$

[50] When $\alpha = 0$, nominal wages are completely rigid and always rise at rate π^R. Hence, inflation is always equal to π^R and cannot be affected by anything.

This shows that, if $\varepsilon_u > \varepsilon_\gamma$ and if a rise in total factor productivity is deflationary, then it must reduce both consumption c and investment δkL. Differentiating (E4), we have:

$$\begin{aligned}
\frac{dl}{d\xi} &= \frac{1}{v''(l)}\left[\tilde{f}(k)\,u'(c) - \xi k f''(k)\,u'(c)\,\frac{dk}{d\xi} + \xi \tilde{f}(k)\,u''(c)\,\frac{dc}{d\xi}\right], \\
&= \frac{\tilde{f}(k)\,u'(c)}{v''(l)} + \frac{kf'(k)\,u'(c)}{v''(l)} \\
&\qquad + \frac{1}{v''(l)}\left[ku'(c) + \frac{\xi\tilde{f}(k)\,u''(c)}{\varepsilon_u - \varepsilon_\gamma}\left[\frac{cu'(c)}{\gamma'(kL)} - \varepsilon_\gamma \frac{L\tilde{f}(k)}{kf''(k)}\right]\right]\frac{d\pi}{d\xi}, \\
&= \frac{f(k)\,u'(c)}{v''(l)} + \frac{1}{v''(l)}\left[ku'(c) + \frac{\xi\tilde{f}(k)\,u''(c)}{\varepsilon_u - \varepsilon_\gamma}\left[\frac{cu'(c)}{\gamma'(kL)} - \varepsilon_\gamma \frac{L\tilde{f}(k)}{kf''(k)}\right]\right]\frac{d\pi}{d\xi}.
\end{aligned}$$

Totally differentiating (E3) yields:

$$\tilde{f}(k)\,\frac{d\pi}{d\xi} - \left[\pi - \left(\pi^R - \alpha\right)\right] kf''(k)\,\frac{dk}{d\xi} = \frac{\alpha}{l}\tilde{f}'\left(\frac{L}{l}k\right)\left[L\frac{dk}{d\xi} + k\frac{dL}{d\xi} - \frac{kL}{l}\frac{dl}{d\xi}\right].$$

Substituting the above expressions into this equation gives:

$$\begin{aligned}
&\tilde{f}(k)\,\frac{d\pi}{d\xi} + \left[\pi - \left(\pi^R - \alpha\right)\right]\frac{k}{\xi}\frac{d\pi}{d\xi} + \left[\pi - \left(\pi^R - \alpha\right)\right]\frac{kf'(k)}{\xi} \\
&\qquad = -\alpha\tilde{f}'\left(\frac{L}{l}k\right)\frac{Lk}{l^2}\frac{f(k)\,u'(c)}{v''(l)} + \frac{\alpha}{l}\tilde{f}'\left(\frac{L}{l}k\right)[\ldots]\,\frac{d\pi}{d\xi}.
\end{aligned}$$

As α tends to zero, $\pi - \left(\pi^R - \alpha\right)$ also tends to zero. Thus, solving for $d\pi/d\xi$ shows that it tends to:

$$\frac{-1}{\tilde{f}(k)}\left[\alpha\tilde{f}'\left(\frac{L}{l}k\right)\frac{Lk}{l^2}\frac{f(k)\,u'(c)}{v''(l)} + \left[\pi - \left(\pi^R - \alpha\right)\right]\frac{kf'(k)}{\xi}\right],$$

which is strictly negative provided that $\alpha > 0$. Thus, for α sufficiently small but strictly positive, we have $d\pi/d\xi < 0$. It follows that a rise in total factor productivity reduces inflation, consumption, investment, and, hence, output.

E.4 Proof of Proposition 4

Totally differentiating (E2) and (E5) with respect to g yields:

$$\begin{aligned}
\frac{dk}{dg} &= \frac{1}{-f''(k)}\frac{d\pi}{dg}, \\
\frac{dL}{dg} &= \frac{1}{f(k)-\delta k}\left[1+\frac{dc}{dg}-L\left[f'(k)-\delta\right]\frac{dk}{dg}\right], \\
&= \frac{L}{c+g}\left[1+\frac{dc}{dg}+L\frac{f'(k)-\delta}{f''(k)}\frac{d\pi}{dg}\right].
\end{aligned}$$

It follows that:

$$\begin{aligned}
k\frac{dL}{dg}+L\frac{dk}{dg} &= \frac{kL}{c+g}+\frac{kL}{c+g}\frac{dc}{dg}+\left[\frac{-L}{f''(k)}+\frac{kL^2\left[f'(k)-\delta\right]}{(c+g)\,f''(k)}\right]\frac{d\pi}{dg}, \\
&= \frac{kL}{c+g}+\frac{kL}{c+g}\frac{dc}{dg}-\frac{L^2}{(c+g)\,f''(k)}\left[\left[f(k)-\delta k\right]-k\left[f'(k)-\delta\right]\right]\frac{d\pi}{dg}, \\
&= \frac{kL}{c+g}+\frac{kL}{c+g}\frac{dc}{dg}-\frac{L^2\tilde{f}(k)}{(c+g)\,f''(k)}\frac{d\pi}{dg}.
\end{aligned}$$

Totally differentiating (E1) yields:

$$\begin{aligned}
\frac{dc}{dg} &= \frac{1}{u''(c)}\left[\frac{-\gamma'(kL)}{(\rho+\pi)^2}\frac{d\pi}{dg}+\frac{\gamma''(kL)}{\rho+\pi}\left[k\frac{dL}{dg}+L\frac{dk}{dg}\right]\right], \\
&= \frac{-\gamma'(kL)}{cu''(c)\,(\rho+\pi)}\left[\frac{cu'(c)}{\gamma'(kL)}\frac{d\pi}{dg}-\frac{\gamma''(kL)}{\gamma'(kL)}\left[\frac{kLc}{c+g}+\frac{kLc}{c+g}\frac{dc}{dg}-\frac{L^2c\tilde{f}(k)}{(c+g)\,f''(k)}\frac{d\pi}{dg}\right]\right], \\
&= \frac{\varepsilon_\gamma}{\varepsilon_u}\frac{c}{c+g}+\frac{1}{\varepsilon_u}\left[\frac{cu'(c)}{\gamma'(kL)}-\varepsilon_\gamma\frac{L\tilde{f}(k)}{kf''(k)}\frac{c}{c+g}\right]\frac{d\pi}{dg}+\frac{\varepsilon_\gamma}{\varepsilon_u}\frac{c}{c+g}\frac{dc}{dg},
\end{aligned}$$

where I have used the fact that $\gamma'(kL)\,/\,(\rho+\pi)=u'(c)$. Hence:

$$\frac{dc}{dg}=\frac{c\varepsilon_\gamma}{(c+g)\,\varepsilon_u-c\varepsilon_\gamma}+\frac{c}{(c+g)\,\varepsilon_u-c\varepsilon_\gamma}\left[\frac{(c+g)\,u'(c)}{\gamma'(kL)}-\varepsilon_\gamma\frac{L\tilde{f}(k)}{kf''(k)}\right]\frac{d\pi}{dg}.$$

It follows that:

$$
\begin{aligned}
k\frac{dL}{dg}+L\frac{dk}{dg} &= \frac{kL}{c+g}\frac{(c+g)\,\varepsilon_u}{(c+g)\,\varepsilon_u-c\varepsilon_\gamma} \\
&\qquad +\frac{kL}{c+g}\frac{1}{(c+g)\,\varepsilon_u-c\varepsilon_\gamma}\left[\frac{c\,(c+g)\,u'(c)}{\gamma'(kL)}-c\varepsilon_\gamma\frac{L\tilde{f}(k)}{kf''(k)}\right. \\
&\qquad\qquad \left.-\left((c+g)\,\varepsilon_u-c\varepsilon_\gamma\right)\frac{L\tilde{f}(k)}{kf''(k)}\right]\frac{d\pi}{dg}, \\
&= \frac{kL\varepsilon_u}{(c+g)\,\varepsilon_u-c\varepsilon_\gamma}+\frac{kL}{(c+g)\,\varepsilon_u-c\varepsilon_\gamma}\left[\frac{cu'(c)}{\gamma'(kL)}-\varepsilon_u\frac{L\tilde{f}(k)}{kf''(k)}\right]\frac{d\pi}{dg}.
\end{aligned}
$$

Differentiating (E4) yields:

$$
\begin{aligned}
\frac{dl}{dg} &= \frac{1}{v''(l)}\left[-kf''(k)\,u'(c)\,\frac{dk}{dg}+\tilde{f}(k)\,u''(c)\,\frac{dc}{dg}\right], \\
&= \frac{c\varepsilon_\gamma}{(c+g)\,\varepsilon_u-c\varepsilon_\gamma}\frac{\tilde{f}(k)\,u''(c)}{v''(l)} \\
&\quad +\frac{1}{v''(l)}\left[ku'(c)+\frac{c\tilde{f}(k)\,u''(c)}{(c+g)\,\varepsilon_u-c\varepsilon_\gamma}\left[\frac{(c+g)\,u'(c)}{\gamma'(kL)}-\varepsilon_\gamma\frac{L\tilde{f}(k)}{kf''(k)}\right]\right]\frac{d\pi}{dg},
\end{aligned}
$$

By totally differentiating (E3), we obtain:

$$
\tilde{f}(k)\,\frac{d\pi}{dg}-\left[\pi-\left(\pi^R-\alpha\right)\right]kf''(k)\,\frac{dk}{dg}=\frac{\alpha}{l}\tilde{f}'\left(\frac{L}{l}k\right)\left[L\frac{dk}{dg}+k\frac{dL}{dg}-\frac{kL}{l}\frac{dl}{dg}\right].
$$

Substituting the above expressions into this equation gives:

$$
\begin{aligned}
&\tilde{f}(k)\,\frac{d\pi}{dg}+\left[\pi-\left(\pi^R-\alpha\right)\right]k\frac{d\pi}{dg} \\
&=\frac{\alpha}{l}\tilde{f}'\left(\frac{L}{l}k\right)\left[\frac{kL\varepsilon_u}{(c+g)\,\varepsilon_u-c\varepsilon_\gamma}-\frac{kL}{l}\frac{c\varepsilon_\gamma}{(c+g)\,\varepsilon_u-c\varepsilon_\gamma}\frac{\tilde{f}(k)\,u''(c)}{v''(l)}\right]+\frac{\alpha}{l}\tilde{f}'\left(\frac{L}{l}k\right)[...]\,\frac{d\pi}{dg}.
\end{aligned}
$$

As α tends to zero, $\pi-\left(\pi^R-\alpha\right)$ also tends to zero. Thus, solving for $d\pi/dg$ shows that it tends to:

$$
\frac{1}{\tilde{f}(k)}\frac{\alpha}{l}\tilde{f}'\left(\frac{L}{l}k\right)\frac{kL}{(c+g)\,\varepsilon_u-c\varepsilon_\gamma}\left[\varepsilon_u-c\varepsilon_\gamma\frac{\tilde{f}(k)\,u''(c)}{lv''(l)}\right].
$$

This shows that $d\pi/dg$ tends to zero as α tends to zero. Moreover, if $(c+g)\,\varepsilon_u>c\varepsilon_\gamma$, then, for α strictly positive but sufficiently small, we have $d\pi/dg>0$.

Finally, the fiscal multiplier is obtained by differentiating (E6):

$$
\begin{aligned}
\frac{dy}{dg} &= 1+\frac{dc}{dg}+\delta\left[k\frac{dL}{dg}+L\frac{dk}{dg}\right], \\
&= 1+\frac{\delta kL\varepsilon_u+c\varepsilon_\gamma}{(c+g)\,\varepsilon_u-c\varepsilon_\gamma}+\frac{1}{(c+g)\,\varepsilon_u-c\varepsilon_\gamma}\left[\frac{c\,(c+g)\,u'(c)}{\gamma'(kL)}-c\varepsilon_\gamma\frac{L\tilde{f}(k)}{kf''(k)}\right. \\
&\qquad\qquad\left.+\frac{c\delta kLu'(c)}{\gamma'(kL)}-\varepsilon_u\frac{\delta L^2\tilde{f}(k)}{f''(k)}\right]\frac{d\pi}{dg}, \\
&= 1+\frac{\delta kL+c\varepsilon_\gamma/\varepsilon_u}{(c+g)-c\varepsilon_\gamma/\varepsilon_u} \\
&\qquad\qquad+\frac{1}{(c+g)\,\varepsilon_u-c\varepsilon_\gamma}\left[\frac{cu'(c)\,f(k)}{\gamma'(kL)}-(\delta kL\varepsilon_u+c\varepsilon_\gamma)\,\frac{L\tilde{f}(k)}{kf''(k)}\right]\frac{d\pi}{dg}.
\end{aligned}
$$

Thus, in the limit as α tends to zero, $d\pi/dg=0$ and the multiplier is equal to:

$$
1+\frac{\delta kL+c\frac{\varepsilon_\gamma}{\varepsilon_u}}{(c+g)-c\frac{\varepsilon_\gamma}{\varepsilon_u}}.
$$

When α is strictly positive, but sufficiently small, then $d\pi/dg>0$ and the multiplier is even larger.

E.5 Proof of Proposition 5

In steady state, welfare is equal to:

$$
W=\frac{u(c)-v(L)+h(\bar{m})+\gamma(kL)}{\rho},
$$

where I have used the fact that, in equilibrium, the wealth of households net of government liabilities must be equal to the stock of physical capital. In the limit as α tends to zero, inflation is equal to π^R. Thus, by (E2) with $\pi=\pi^R$, we know that k is independent of the fiscal policy. We have:

$$\begin{aligned}
\frac{dW}{dg} &= \frac{1}{\rho}\left[u'(c)\frac{dc}{dg} + \left[k\gamma'(kL) - v'(L)\right]\frac{dL}{dg}\right], \\
&= \frac{1}{\rho}\left[u'(c)\frac{c\varepsilon_\gamma}{(c+g)\varepsilon_u - c\varepsilon_\gamma} + \left[k\gamma'(kL) - v'(L)\right]\frac{L\varepsilon_u}{(c+g)\varepsilon_u - c\varepsilon_\gamma}\right], \\
&= \frac{1}{\rho}\frac{1}{(c+g)\varepsilon_u - c\varepsilon_\gamma}\left[cu'(c)\varepsilon_\gamma + \left[kL\left(\rho+\pi^R\right)u'(c) - Lv'(L)\right]\varepsilon_u\right], \\
&= \frac{1}{\rho}\frac{1}{(c+g)\varepsilon_u - c\varepsilon_\gamma}\left[c\frac{v'(l)}{\tilde{f}(k)}\varepsilon_\gamma + \left[kL\left(\rho+\pi^R\right) - \tilde{f}(k)L\frac{v'(L)}{v'(l)}\right]\frac{v'(l)}{\tilde{f}(k)}\varepsilon_u\right], \\
&= \frac{1}{\rho}\frac{\varepsilon_u}{(c+g)\varepsilon_u - c\varepsilon_\gamma}\frac{v'(l)}{\tilde{f}(k)}\left[c\frac{\varepsilon_\gamma}{\varepsilon_u} + kL\left(\rho+\pi^R\right) - \left[Lf(k) - kL\left(\delta-\pi^R\right)\right]\frac{v'(L)}{v'(l)}\right], \\
&= \frac{1}{\rho}\frac{\varepsilon_u}{(c+g)\varepsilon_u - c\varepsilon_\gamma}\frac{v'(l)}{\tilde{f}(k)}\left[c\frac{\varepsilon_\gamma}{\varepsilon_u} + kL\left[\rho+\pi^R\left(1-\frac{v'(L)}{v'(l)}\right)\right] - [c+g]\frac{v'(L)}{v'(l)}\right],
\end{aligned}$$

where, to get the second line, I have used results from the proof of Proposition 4 with $d\pi/dg = 0$. If $(c+g)\varepsilon_u > c\varepsilon_\gamma$, then, in the limit as α tends to zero, an increase in g is welfare enhancing if and only if the term in the main square bracket is positive.

F Proof of Proposition 8

The steady state equilibrium (c, K, π, l, L) under the optimal policy with inflation target $\hat{\pi} \in \left[\pi^R, \pi^*\right]$ is fully characterized by:

$$\begin{gathered}
u'(c) = \frac{\gamma'(K)}{\rho + \pi - r^n - \hat{\pi}}, \\
\left(1 - s^I\right)\left[\hat{\pi} - \pi\right] = \left(1 - \tau^K\right)\left[F_K(K,L) - \delta - r^n\right], \\
v'(l) = F_L(K,L)\,u'(c), \\
F(K,L) = c + \delta K, \\
F_L(K,L) = \begin{cases} F_L(K,l) & \text{if } \pi \in \left[\pi^R, +\infty\right) \\ \frac{\alpha}{\pi - (\pi^R - \alpha)}F_L(K,l) & \text{if } \pi \in \left(\pi^R - \alpha, \pi^R\right) \end{cases},
\end{gathered}$$

where I have substituted the optimal wealth tax (78) into the demand for consumption (61) to obtain the first line and I have substituted the optimal corporate income tax policy (79) into the demand for investment resulting from (57) and (65) to obtain the second line.[51]

When $\pi = \hat{\pi} \in \left[\pi^R, \pi^*\right]$, the above equations that characterize the steady state

[51] I have used the fact that, in steady state, the nominal interest rate must be equal to zero. This must be the case in the secular stagnation steady state as $\phi = \hat{\pi} \geq \pi^R$ (cf. footnote 14). The frictionless steady state is such that $r = -\hat{\pi}$, which implies that $i = r + \pi = -\hat{\pi} + \phi = 0$.

equilibrium of the decentralized economy coincide with those that characterize the first-best allocation. Hence, there is a steady state equilibrium with $\pi = \hat{\pi}$ that implements the first-best allocation (as we already know from Proposition 7). However, proceeding as in the proof of Lemma 3, it can be shown that a secular stagnation steady state equilibrium with $\pi \in \left(\pi^R - \alpha, \pi^R\right)$ also exists provided that $c > 0$ when $\pi = \pi^R - \alpha$, i.e. provided that $\rho + \left(\pi^R - \alpha\right) - r^n - \hat{\pi} > 0$ or equivalently $\alpha < \rho + \pi^R + \bar{\pi} - \hat{\pi}$, where I have used the fact that $r^n = -\bar{\pi}$.[52] Clearly, if $\hat{\pi} > \pi^R$, then the secular stagnation equilibrium cannot coincide with the frictionless equilibrium.

Let us now prove that, conversely, the two equilibria can coincide when $\hat{\pi} = \pi^R$. The frictionless and secular stagnation equilibria are characterized by the same set of equations, except for one exception: the frictionless equilibrium satisfies $L = l$, while the secular stagnation equilibrium satisfies $\left[\pi - \left(\pi^R - \alpha\right)\right] F_L(K, L) = \alpha F_L(K, l)$. Clearly, when $\hat{\pi} = \pi^R$, the frictionless equilibrium with $\pi = \hat{\pi} = \pi^R$ satisfies all the equations that characterize the secular stagnation equilibrium, including $\left[\pi - \left(\pi^R - \alpha\right)\right] F_L(K, L) = \alpha F_L(K, l)$.[53] Hence, when $\hat{\pi} = \pi^R$, there is a secular stagnation equilibrium that coincides with the frictionless equilibrium. If the secular stagnation equilibrium is unique, then it must coincide with the frictionless equilibrium.

Proceeding as in the proof of Lemma 3, it can be shown that, if α is sufficiently small, the secular stagnation steady state equilibrium is unique.[54]

[52] As $\bar{\pi} > \pi^* \geq \hat{\pi}$, the optimal policy increases the range of values of α for which a secular stagnation equilibrium exists.

[53] The secular stagnation equilibrium has so far been defined for $\pi \in \left(\pi^R - \alpha, \pi^R\right)$, rather than $\pi \in \left(\pi^R - \alpha, \pi^R\right]$, since it typically satisfies $L < l$, which requires $\pi < \pi^R$. However, it must fundamentally be characterized by the first four equations of this proof together with $\left[\pi - \left(\pi^R - \alpha\right)\right] F_L(K, L) = \alpha F_L(K, l)$. When the optimal policy with $\hat{\pi} = \pi^R$ is implemented, it turns out that the secular stagnation equilibrium is consistent with $L = l$ and, hence, with $\pi = \pi^R$.

[54] Alternatively, uniqueness can be shown by proving that, for α sufficiently small, any secular stagnation steady state must satisfy $dy^s/d\pi > dy^d/d\pi$, where $y^s = F(K, L)$ and $y^d = c + \delta K$.