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The Dugdale model was initially developed in the case of a mode I loading. It was extended to other modes and to the mixed mode case. The exact solutions were given for all these modes in the case of an infinite medium with a straight crack. This work is an application of the Dugdale model to a crack in a semi infinite structure submitted to a mixed mode loading. The coupled system of singular integral equations of the first kind corresponding to the elastostatic problem is solved semi-analytically. Particular attention is needed in the resolution because of jump discontinuities in the loading of the crack faces. The criteria of propagation are deduced from the revisited Griffith theory (G. Francfort, J-J. Marigo, Journal of Mechanics and Physics of Solids (1998) 46:8 1319–1342). The presented results show the evolution of the applied load and critical stress with the crack length. The shape of the crack gap is also presented. A comparison with the problem of a crack in an infinite structure is performed.

1. Introduction

The exact solution for a straight Dugdale crack in an infinite medium was established by several authors. It was initially given in the case of a mode I loading (see for example Bui (1978) and Becker and Gross (1987)). Ref. Becker and Gross (1987) also presented the solution in the case of mode II and mixed mode loadings. In mode III, the exact solution was given by Nicholson (1993). The aim of the present work, is the deduction of an approximate solution of a Dugdale crack in a semi infinite structure under a mixed mode loading.

The Dugdale model was initially established in the field of elastic plastic fracture mechanics (Dugdale, 1960). In this paper, it is interpreted as a particular case of the cohesive zone or the Barenblatt model (Barenblatt, 1962; Needleman, 1987; Elices et al., 2002; Paggi and Wriggers, 2011) under the framework of brittle fracture mechanics. The crack propagation criteria are deduced using the revisited Griffith theory (Francfort and Marigo, 1998).

The paper is organized as follows. Both general cohesive zone and Dugdale models in a mixed mode case are presented in section 2. The variational formulation is also included. In section 3, the studied structure is depicted and the crack propagation criteria established. In section 4, the system of singular integral equations is presented, and the resolution method exposed. Section 5 is devoted to numerical results consisting in a parametric study of the problem.

2. The cohesive zone model in the mixed mode case

Throughout the paper, all the analysis is made in the plane elasticity setting. One uses a cartesian system (x_1,x_2,x_3) with its canonical orthonormal basis $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$. Consider a body, the reference configuration of which is the open subset Ω of $\mathbb{R}^2$ in the plane (x_1,x_2). The loading consists in the prescribed displacement $\mathbf{U}$, parallel to the (x_1,x_2) plane, on the part $\partial\Omega_d$ of the boundary, and prescribed surface forces $\mathbf{F}$, parallel to the (x_1,x_2) plane, on the complementary part $\partial\Omega_f$ of the boundary and body forces $\mathbf{f}$ in the same plane. All these data are supposed smooth. The loading causes the propagation of a crack along a smooth simple predefined path Γ with unit normal $\mathbf{n}$ and across which the displacement can be discontinuous. In the uncracked part $\Omega\backslash\Gamma$ of the body, the material has an isotropic linear elastic behavior characterized by stiffness tensor A.

Let $[\![u_n]\!]$ and $[\![u_t]\!]$ respectively be the jump of the normal and tangential displacements at a point of the crack path, called *gaps*. Also, let σ and τ respectively be the normal and tangential

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components of the stress vector on Γ, called *cohesive forces*. The relationships between *cohesive forces* and current *gaps* are obtained via a variational approach from fundamental assumptions on the surface energy density.

2.1. Variational formulation

Assume surface energy density ϕ is a function of the gaps $[\![u_n]\!]$ and $|[\![u_t]\!]|$. In order to obtain precise mathematical results, it is necessary to suppose that ϕ enjoys some relevant concave and monotonic properties (see Marigo and Truskinovsky, 2004). However, since in this paper all the developments are made with Dugdale's surface energy, it is simply assumed that ϕ is monotonically increasing and piecewise smooth with $\phi(0{,}0) = 0$, $\sigma_c = (\partial\phi/\partial[\![u_n]\!])(0,\bullet) > 0$ and $\tau_c = \partial\phi/\partial|[\![u_t]\!]|(\bullet,0) > 0$. σ_c and τ_c are called *critical stresses*.

Let $\mathbf{v}$ be a kinematically admissible displacement, *i.e.*

$$\mathbf{v}\in\mathcal{C} = \left\{\mathbf{v}\in\mathbf{H}^1\left(\boldsymbol{\Omega}\backslash\boldsymbol{\Gamma},\mathbb{R}^2\right) : \mathbf{v} = \mathbf{U} \text{ on } \partial\boldsymbol{\Omega}_\mathbf{d}\right\}$$

where H^1 denotes the usual Sobolev space. The associated total energy of the body is given by

$$\mathcal{E}(v) = \frac{1}{2}\int_{\Omega\backslash\Gamma} \mathbf{A}\varepsilon(v)\cdot\varepsilon(v)\mathrm{d}x + \int_\Gamma \phi([\![v_n]\!],|[\![v_t]\!]|)\mathrm{d}s - \int_{\partial\Omega_f} F\cdot v\mathrm{d}s - \int_{\Omega\backslash\Gamma} f\cdot v\mathrm{d}x, \tag{1}$$

where $\varepsilon(v)$ denotes the linearized strain tensor field, $2\varepsilon_{ij}(v) = v_{i,j} + v_{j,i}$. The true displacement field $\mathbf{u}$ is (the) one in $\mathcal{C}$ which satisfies the following local minimality condition:

$$\forall v\in\mathcal{C},\quad \exists\overline{a}>0 : \forall a\in[0,\overline{a}],\quad \mathcal{E}(u) \le \mathcal{E}(u + a(v-u)). \tag{2}$$

Dividing by $a > 0$ the above inequality and getting to the limit when $a\downarrow 0$, the following so-called *first order optimality condition* is obtained

$$\forall v\in\mathcal{C},\quad \int_{\partial\Omega_f}(\sigma n - F)(v-u)\mathrm{d}s - \int_\Gamma\left(\sigma - \frac{\partial\phi}{\partial[\![u_n]\!]}(\bullet,\bullet)\right) \times [\![(v_n-u_n)]\!]\mathrm{d}s - \int_\Gamma\left(\tau - sign[\![u_t]\!]\frac{\partial\phi}{\partial|[\![u_t]\!]|}(\bullet,\bullet)\right)[\![(v_t-u_t)]\!]\mathrm{d}s - \int_{\Omega\backslash\Gamma}(div\sigma + f)(v-u)\mathrm{d}x \ge 0. \tag{3}$$

The variational inequality (3) is equivalent to a system of local equalities and inequalities which are obtained by considering different types of test fields v.

1. v is chosen such that $[\![v-u]\!] = 0$ on Γ. Inserting into (3) and using standard arguments of Calculus of Variations leads to the local equilibrium equations and the natural boundary conditions

$$div\sigma + f = 0 \quad \text{in} \quad \Omega\backslash\Gamma,\quad \sigma n = F \quad \text{on} \quad \partial\Omega_f, \tag{4}$$

2. After inserting (4) into (3), the first order optimal condition becomes

$$\forall v\in\mathcal{C},\quad \int_{\Gamma_d}\left(\sigma - \frac{\partial\phi}{\partial[\![u_n]\!]}(\bullet,\bullet)\right)[\![v_n-u_n]\!]\mathrm{d}s + \int_{\Gamma_b}(\sigma-\sigma_c)[\![v_n]\!]\mathrm{d}s + \int_{\Gamma_d^+}\left(\tau - \frac{\partial\phi}{\partial|[\![u_t]\!]|}(\bullet,\bullet)\right)[\![v_t-u_t]\!]\mathrm{d}s + \int_{\Gamma_d^-}\left(\tau + \frac{\partial\phi}{\partial|[\![u_t]\!]|}(\bullet,\bullet)\right)[\![v_t-u_t]\!]\mathrm{d}s + \int_{\Gamma_b}(\tau[\![v_t]\!] - \tau_c|[\![v_t]\!]|)\mathrm{d}s \le 0, \tag{5}$$

where the crack path Γ is subdivided into two parts:

- Γ_b is the bonded part where the *gaps* are nil.
- Γ_d is the debonded part where the *gaps* are nonzero.

Γ_d is also subdivided into two parts:

- Γ_d^+ where $[\![u_t]\!] > 0$,
- Γ_d^- where $[\![u_t]\!] < 0$.

Remark 1.

- *Positive values of normal gap $[\![u_n]\!]$ are always assumed, i.e., there is no contact at crack faces.*
- *In the deduction of equation (5) from (3), the continuity of the traction vector has been assumed across Γ for equilibrium reasons. The case of possible jumps in tractions on imperfect interfaces (see Gu and He, 2011) has not been considered. In other words it is the case of a perfect interface.*

3. Inequality (5) is verified if and only if cohesive force repartition verifies

$$\begin{cases} |\tau| \le \tau_c & \text{on} \quad \Gamma_b \\ \sigma \le \sigma_c & \text{on} \quad \Gamma_b \\ \tau = \dfrac{\partial\phi}{\partial|[\![u_t]\!]|}(\bullet,\bullet) & \text{on} \quad \Gamma_d^+ \\ \tau = -\dfrac{\partial\phi}{\partial|[\![u_t]\!]|}(\bullet,\bullet) & \text{on} \quad \Gamma_d^- \\ \sigma = \dfrac{\partial\phi}{\partial[\![u_n]\!]}(\bullet,\bullet) & \text{on} \quad \Gamma_d \end{cases} \tag{6}$$

Remark 2. *The first order optimality condition contains not only the stress-gap relation (the three last equations of (6)) but also the stress debonding criterion for the crack onset (the two first equations of (6)).*

2.2. The Dugdale model in the mixed mode case

The Dugdale model formulated in energetic terms was presented in Ferdjani et al. (2007) in the mode I case. Below, a generalization in the mixed mode case can be found. The surface energy density is defined on $[0,+\infty)\cup(-\infty,+\infty)$ by:

$$\phi([\![u_n]\!],|[\![u_t]\!]|) = min(\sigma_c[\![u_n]\!] + \tau_c|[\![u_t]\!]|, G_c), \tag{7}$$

where G_c is the critical energy release rate of the Griffith theory.Let the previous formulation and results be particularized to the case of Dugdale's model. Introducing (7) into (6) gives:

$$\begin{cases} \tau = \begin{cases} 0 & \text{on} \quad \Gamma_d^+ \quad \text{if} \quad \sigma_c[\![u_n]\!] + \tau_c\left|[\![u_t]\!]\right| \geq G_c \\ \tau_c & \text{on} \quad \Gamma_d^+ \quad \text{if} \quad \sigma_c[\![u_n]\!] + \tau_c\left|[\![u_t]\!]\right| < G_c \end{cases} \\ \tau = \begin{cases} 0 & \text{on} \quad \Gamma_d^- \quad \text{if} \quad \sigma_c[\![u_n]\!] + \tau_c\left|[\![u_t]\!]\right| \geq G_c \\ -\tau_c & \text{on} \quad \Gamma_d^- \quad \text{if} \quad \sigma_c[\![u_n]\!] + \tau_c\left|[\![u_t]\!]\right| < G_c \end{cases} \\ \sigma = \begin{cases} 0 & \text{on} \quad \Gamma_d \quad \text{if} \quad \sigma_c[\![u_n]\!] + \tau_c|[\![u_t]\!]| \geq G_c \\ \sigma_c & \text{on} \quad \Gamma_d \quad \text{if} \quad \sigma_c[\![u_n]\!] + \tau_c|[\![u_t]\!]| < G_c \end{cases} \end{cases} \tag{8}$$

It may be observed from (8) that the cohesive force vanishes as soon as $\sigma_c[\![u_n]\!] + \tau_c|[\![u_t]\!]| \geq G_c$. Consequently, the debonded part or the *created crack* Γ_d is divided into two zones: the *process zone* Γ_c where $\sigma_c[\![u_n]\!] + \tau_c|[\![u_t]\!]| < G_c$ and the *traction free crack* Γ_0 where $\sigma_c[\![u_n]\!] + \tau_c|[\![u_t]\!]| > G_c$.

3. The studied structure and crack propagation criteria

Consider a semi infinite body Ω constituted by an infinite strip $(-\infty,+\infty) \times (-h,0)$ bonded to a half-plane $(-\infty,+\infty) \times (0,+\infty)$. An initial crack $\mathcal{D} = [-\ell_0, \ell_0] \times \{0\}$ of length $2\ell_0$ exists at the interface (Fig. 1). The strip and half-plane are made of the same isotropic material the elastic properties of which are characterized by stiffness tensor A. The crack faces are submitted to a uniform pressure p_0, increasing from zero, and the body forces are neglected. The edge of the half plane ($x_2 = -h$) is free from stress. By superposition, the problem is equivalent to the case of uniform tensile stresses p_0 applied at $x_2 = h$ and $x_2 \to -\infty$. The brittle fracture of the interface is modelized with the Dugdale model.

3.1. Onset and crack propagation

Since the critical stresses of the material are higher than those of the interface, assume the crack propagates horizontally along the interface. Moreover, for reasons of symmetry, assume the crack propagates along the axis $x_2 = 0$ in a symmetrical manner from the points $(\pm\ell_0, 0)$. Let Γ_d be the created crack and $x_1 = \pm\ell_a$ the position of its tips (Fig. 1):

$$\Gamma_d = [-\ell_a, -\ell_0] \times \{0\} \cup [+\ell_0, +\ell_a] \times \{0\}$$

It has previously been seen (paragraph 2.2) that the crack faces $(-\ell_a, -\ell_0)$ and $(+\ell_0, +\ell_a)$ (of the axis $x_2 = 0$) can be divided into two parts:

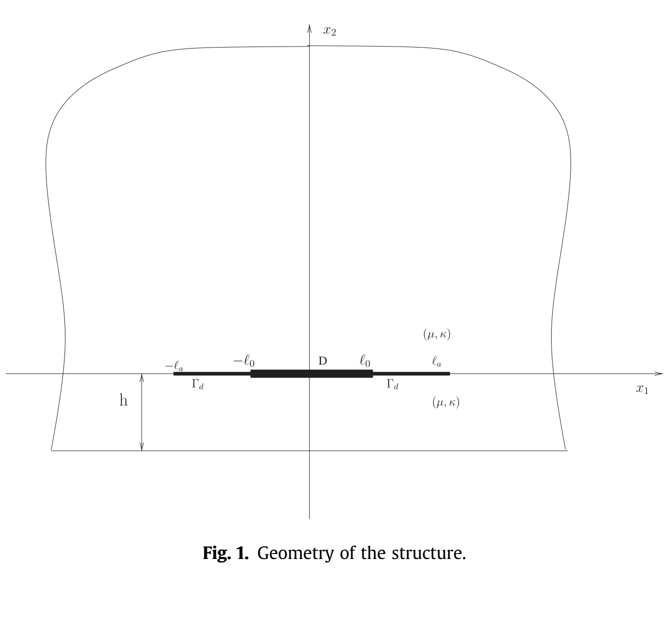

Fig. 1. Geometry of the structure.

- The first, close to the crack tip and named the process zone, is subjected to the constant normal cohesive force σ_c and shearing cohesive force $\pm\tau_c$.
- The second, named traction free crack, is close to the initial crack without cohesive force.

These two zones are separated by the limit points $x_1 = \pm\ell_c$. Noting that, the values of ℓ_a and ℓ_c depend on the value of the remote loading p_0 with assumption $\ell_a \geq \ell_c \geq \ell_0$. At the beginning of loading, the initial conditions are: $\ell_a = \ell_c = \ell_0$.

In the present case, the crack growth follows two phases: the cohesive phase and the propagation phase. The different criteria of the initiation and the propagation of these zones are studied in the following sections.

3.1.1. Cohesive crack phase $0 < p_0 < p_r$

When $p_0 \neq 0$, a crack must appear in a manner such that the maximal normal stress (shearing stress) on the interface remains less than the critical value $\sigma_c\,(\tau_c)$. When the load is sufficiently close to 0, the length of the crack is sufficiently small so that the quantity $\sigma_c[\![u_n]\!] + \tau_c|[\![u_t]\!]|$ is everywhere smaller than the critical value G_c. Consequently, all the faces of the created crack Γ_d are submitted to cohesive forces of intensity σ_c and $\pm\tau_c$. Depending on the sign of $[\![u_t]\!]$, the tangential cohesive force may be positive or negative. The same problem with a Griffith crack has been considered by Erdogan et al. (1973), the computed values of the stress intensity factor k_2 at the right tip have been found all positive. Consequently, for the considered problem, the tangential traction is positive for $x_1 \in [\ell_0, \ell_a]$ and by symmetry, negative for $x_1 \in [-\ell_a, -\ell_0]$. In summary, in the cohesive phase the faces of the whole crack $\mathcal{D} \cup \Gamma_d$ are submitted to normal and tangential loadings $\sigma(x_1)$ and $\tau(x_1)$ given by:

$$\sigma(x_1) = \begin{cases} -p_0 + \sigma_c & \text{if} \quad \ell_0 < |x_1| < \ell_a \\ -p_0 & \text{if} \quad |x_1| < \ell_0 \end{cases} \tag{9}$$

$$\tau(x_1) = \begin{cases} \tau_c & \text{if} \quad \ell_0 < x_1 < \ell_a \\ -\tau_c & \text{if} \quad -\ell_a < x_1 < -\ell_0 \\ 0 & \text{if} \quad |x_1| < \ell_0 \end{cases} \tag{10}$$

It remains to determine the law governing the evolution of the tips $(\pm\ell_a, 0)$ of the process zone with p_0. This criterion is first stated in terms of the energy release rate and then interpreted in terms of the stress intensity factors. The total energy of the cracked body at equilibrium is a function of the load p_0 and of the position ℓ_a of the tip of the crack. By including the surface energy due to the cohesive forces, the total energy reads as:

$$\mathcal{E}(p_0, \ell_a) = \frac{1}{2}\int_{\Omega\setminus(\mathcal{D}\cup\Gamma_d)} \mathbf{A}\varepsilon(u)\cdot\varepsilon(u)\mathrm{d}x + \int_{\Gamma_d} \sigma_c[\![u_2]\!]\mathrm{d}x_1 - \int_{\mathcal{D}\cup\Gamma_d} p_0[\![u_2]\!]\mathrm{d}x_1 + \int_{\Gamma_d} \tau_c|[\![u_1]\!]|\mathrm{d}x_1. \tag{11}$$

In the spirit of the revised theory of fracture presented first by Francfort and Marigo (1998) in the Griffith context and then by Marigo and Truskinovsky (2004) or Jaubert and Marigo (2006) in the Barenblatt context of surface energy assumption, the length of the created crack must be such that the total energy of the body be a local minimum for a given load. Specifically, the local minimum condition reads as:

$$\exists a > 0, \forall \ell_a^* : \ell_a^* \in [\ell_0, \infty), |\ell_a^* - \ell_a| \leq a, \quad \varepsilon(p_0, \ell_a) \leq \varepsilon(p_0, \ell_a^*). \tag{12}$$

Consequently, since it is sought a local minimum ℓ_a lying in the open interval (ℓ_0, ∞), ℓ_a must be a stationary point of $\mathcal{E}(p_0)$ and then satisfy

$$-\frac{\partial \mathcal{E}}{\partial \ell_a}(p_0, \ell_a) = 0 \tag{13}$$

or, in other words, the total energy release rate (*E.R.R.*) due to a growth of the crack must be 0.

This energetic criterion turns out to be a condition of vanishing of the singularity at the crack tip. Indeed, since the cohesive forces are constant (it suffices that they are smooth functions of x_1 in order that the following singularity property holds), they do not change the form of the singularity at the crack tip and the singularity is like that of a traction free crack. In any case, the Irwin formula holds, the *E.R.R.* and the stress intensity factors are related by

$$-\frac{\partial \mathcal{E}}{\partial \ell_a}(p_0, \ell_a) = \begin{cases} 2\pi \dfrac{1-\nu^2}{E}\left(k_1^2 + k_2^2\right) & \text{for plane strain} \\ 2\pi \dfrac{1}{E}\left(k_1^2 + k_2^2\right) & \text{for plane stress} \end{cases}, \tag{14}$$

the factor 2 being due to the presence of two tips. In (14) E and ν are the Young modulus and the Poisson coefficient of the material, k_1 and k_2 are the mode I and II *S.I.F.* respectively. So, criterion (13) is equivalent to $k_1 = k_2 = 0$ meaning that the length of the process zone must be such that there does not exist any singularity at the crack tip. This conforms to the Barenblatt idea that cohesive forces are present to correct the non physical property of infinite stresses induced by the Griffith theory. This phase finishes when the quantity $\sigma_c[\![u_n]\!] + \tau_c|[\![u_t]\!]|$ at $x_1 = \pm\ell_0$ reaches the critical value G_c. That means that a traction free crack must appear. The corresponding value of the load will be called the *rupture* load and is then defined by

$$p_r = \sup\{p_0 > 0 : \sigma_c[\![u_n]\!](\pm\ell_0) + \tau_c|[\![u_t]\!](\pm\ell_0)| < G_c\}. \tag{15}$$

3.1.2. *Propagation phase*

If the load is increased beyond p_r, then an equilibrium state cannot be found without considering the initiation and the propagation of a stress free part on the faces of the created crack. Thus, the crack Γ_d should be divided into two parts, the process zone Γ_c and the traction free crack Γ_0. Denoting by ℓ_c and ℓ_a their respective tips, gives

$$\begin{aligned} &\Omega_c^0 = \Omega\backslash(\mathcal{D}\cup\Gamma_d), \quad \Gamma_d = \Gamma_0\cup\Gamma_c \\ &\Gamma_c = (-\ell_a, -\ell_c] \times \{0\}\cup[\ell_c, \ell_a) \times \{0\}, \\ &\Gamma_0 = (-\ell_c, -\ell_0] \times \{0\}\cup[\ell_0, \ell_c) \times \{0\}. \end{aligned} \tag{16}$$

In the propagation phase the faces of the whole crack $\mathcal{D}\cup\Gamma_d$ are submitted to normal and tangential loadings $\sigma(x_1)$ and $\tau(x_1)$ given by:

$$\sigma(x_1) = \begin{cases} -p_0 + \sigma_c & \text{if} \quad \ell_c < |x_1| < \ell_a \\ -p_0 & \text{if} \quad |x_1| < \ell_c \end{cases} \tag{17}$$

$$\tau(x_1) = \begin{cases} \tau_c & \text{if} \quad \ell_c < x_1 < \ell_a \\ -\tau_c & \text{if} \quad -\ell_a < x_1 < -\ell_c \\ 0 & \text{if} \quad |x_1| < \ell_c \end{cases} \tag{18}$$

This stress field σ satisfies the following variational equation (Theorem of Virtual Work, see (Gurtin, 1981)): For any "smooth" vector field on Ω_c^0,

$$\begin{aligned} &\int_{\Omega_c^0} \sigma\cdot\varepsilon(v)\mathrm{d}x - \int_{\mathcal{D}\cup\Gamma_d} p_0[\![v_2]\!]\mathrm{d}x_1 + \int_{\Gamma_c} \sigma_c[\![v_2]\!]\mathrm{d}x_1 \\ &+ \int_{\Gamma_c} \tau_c|[\![v_1]\!]|\mathrm{d}x_1 = 0. \end{aligned} \tag{19}$$

It remains to define the laws governing the evolution of the tips ℓ_c and ℓ_a. As in the case of the cohesive phase, these laws are first stated by using energetic arguments. The total energy of the body at equilibrium is given by

$$\begin{aligned} \mathcal{E}(p_0, \ell_c, \ell_a) = &\frac{1}{2}\int_{\Omega_c^0} \mathbf{A}\varepsilon(u)\cdot\varepsilon(u)\mathrm{d}x + \int_{\Gamma_c} \sigma_c[\![u_2]\!]\mathrm{d}x_1 \\ &- \int_{\mathcal{D}\cup\Gamma_d} p_0[\![u_2]\!]\mathrm{d}x_1 + \int_{\Gamma_c} \tau_c|[\![u_1]\!]|\mathrm{d}x_1 \\ &+ 2G_c(\ell_c - \ell_0). \end{aligned} \tag{20}$$

The evolution of the tips ℓ_c and ℓ_a with the load p_0 must be such that (ℓ_c, ℓ_a) is a local minimum of $\mathcal{E}(p_0, \cdot, \cdot)$ for a given p_0. Specifically, the criterion reads as

$$\begin{aligned} &\exists\, a > 0, \quad \forall\,(\ell_c^*, \ell_a^*) : \ell_0 \le \ell_c^* \le \ell_a^* < \infty, \quad |\ell_c^* - \ell_c| + |\ell_a^* - \ell_a| \le a, \\ &\mathcal{E}(p_0, \ell_c, \ell_a) \le \mathcal{E}(p_0, \ell_c^*, \ell_a^*). \end{aligned} \tag{21}$$

Seeking a local minimum such that $\ell_0 < \ell_c < \ell_a < \infty$, yields the following system of (necessary) conditions linking ℓ_c and ℓ_a to p_0:

$$-\frac{\partial \mathcal{E}}{\partial \ell_a}(p_0, \ell_c, \ell_a) = 0, \quad -\frac{\partial \mathcal{E}}{\partial \ell_c}(p_0, \ell_c, \ell_a) = 0. \tag{22}$$

In other words, the tips of the process zone and the stress free crack must be such that the total energy release rates due to the propagation of one or another tip vanish. Let an interpretation of these criteria in terms of local quantities be given. The condition (22a) is the same as in the cohesive phase. The displacement field is *a priori* singular at the tips $x_1 = \pm\ell_a$. The Irwin formula (14) holds again and then (22a) is still equivalent to the vanishing of the stress intensity factors k_1 and k_2. On the other hand, the displacement field is not singular at the points $x_1 = \pm\ell_c$, because the loading is simply discontinuous at these points. Since the field $\mathbf{u}$ is not singular at the tips $\pm\ell_c$, deriving formally $\mathcal{E}(p_0, \ell_c, \ell_a)$ with respect to ℓ_c under the integration sign leads to

$$\begin{aligned} -\frac{\partial \mathcal{E}}{\partial \ell_c}(p_0, \ell_c, \ell_a) = &-\int_{\Omega_c^0} \mathbf{A}\varepsilon(u)\cdot\varepsilon\left(\frac{\partial u}{\partial \ell_c}\right)\mathrm{d}x - \int_{\Gamma_c} \sigma_c[\![\frac{\partial u_2}{\partial \ell_c}]\!]\mathrm{d}x_1 \\ &+ \int_{\mathcal{D}\cup\Gamma_d} p_0[\![\frac{\partial u_2}{\partial \ell_c}]\!]\mathrm{d}x_1 - \int_{\Gamma_c} \tau_c sign[\![u_1]\!][\![\frac{\partial u_1}{\partial \ell_c}]\!]\mathrm{d}x_1 \\ &+ 2\sigma_c[\![u_2]\!](\pm\ell_c) + 2\tau_c|[\![u_1]\!](\pm\ell_c)| - 2G_c, \end{aligned} \tag{23}$$

with the use of the symmetry of the body. In (23), $\partial u/\partial \ell_c$ represents the rate of the displacement field at equilibrium under the load p_0 due to a (virtual) growth of the traction free crack, the tip of the process zone remaining fixed. By virtue of the variational equation (19), the terms containing $\partial u/\partial \ell_c$ disappear resulting in

$$-\frac{\partial \mathcal{E}}{\partial \ell_c}(p_0, \ell_c, \ell_a) = 2\sigma_c[\![u_2]\!](\pm\ell_c) + 2\tau_c|[\![u_1]\!](\pm\ell_c)| - 2G_c. \tag{24}$$

the factor 2 being due to the presence of two tips. Thus, the criterion of propagation (22b) is equivalent to $\sigma_c[\![u_2]\!](\pm\ell_c)+\tau_c|[\![u_1]\!](\pm\ell_c)|=G_c$. Finally, (22) is equivalent to

$$k_1(\pm\ell_a)=0,\quad k_2(\pm\ell_a)=0,\quad \sigma_c[\![u_2]\!](\pm\ell_c)+\tau_c|[\![u_1]\!](\pm\ell_c)|=G_c. \tag{25}$$

4. System of singular integral equations and solution

This problem has been treated by Erdogan et al. (1973) for the case of a Griffith crack. In this paper, the case differs only in the shape of the load on the crack faces. Thus by defining

$$\begin{aligned}\phi_1(x_1)&=\frac{\partial}{\partial x_1}[u_1(x_1,+0)-u_1(x_1,-0)],\\ \phi_2(x_1)&=\frac{\partial}{\partial x_1}[u_2(x_1,+0)-u_2(x_1,-0)],\end{aligned} \tag{26}$$

and using Fourier transforms, the problem may be formulated as

$$\frac{1}{\pi}\int_{-\ell_a}^{\ell_a}\frac{\phi_i(t)}{t-x_1}\mathrm{d}t+\frac{1}{\pi}\int_{-\ell_a}^{\ell_a}\sum_1^2 k_{ij}(x_1,t)\phi_j(t)\mathrm{d}t=f_i(x_1), \tag{27}$$
$$(i=1,2;|x_1|<\ell_a)$$

with the following compatibility conditions:

$$\int_{-\ell_a}^{\ell_a}\phi_1(t)\mathrm{d}t=0,\qquad \int_{-\ell_a}^{\ell_a}\phi_2(t)\mathrm{d}t=0. \tag{28}$$

The second member of (27) and the Fredholm kernels $k_{ij}(x_1,t),(i,j=1,2)$ are given by

$$\begin{aligned}
f_1(x_1)&=\frac{1+\kappa}{2\mu}\tau(x_1),\quad f_2(x_1)=\frac{1+\kappa}{2\mu}\sigma(x_1),\\
k_{11}(x_1,t)&=-\frac{t-x_1}{(t-x_1)^2+4h^2}+\frac{8h^2(t-x_1)}{\left[(t-x_1)^2+4h^2\right]^2}-\frac{4h^2(t-x_1)\left[12h^2-(t-x_1)^2\right]}{\left[(t-x_1)^2+4h^2\right]^3},\\
k_{12}(x_1,t)&=k_{21}(x_1,t)=-\frac{8h^3\left[-3(t-x_1)^2+4h^2\right]}{\left[(t-x_1)^2+4h^2\right]^3},\\
k_{22}(x_1,t)&=-\frac{t-x_1}{(t-x_1)^2+4h^2}-\frac{8h^2(t-x_1)}{\left[(t-x_1)^2+4h^2\right]^2}-\frac{4h^2(t-x_1)\left[12h^2-(t-x_1)^2\right]}{\left[(t-x_1)^2+4h^2\right]^3}.
\end{aligned} \tag{29}$$

where $\kappa=3-4\nu$ for plane strain and $\kappa=(3-\nu)/(1+\nu)$ for plane stress, μ is the shear modulus.

(27) is a system of singular integral equations of the first kind, the paragraph below presents its resolution.

4.1. *The solution of the system of integral equations*

The solution in the case of the cohesive phase is presented. The resolution method is the same for the propagation phase, changing ℓ_0 by ℓ_c in the equations. Defining the following normalized quantities:

$$r=\frac{x_1}{\ell_a},\quad s=\frac{t}{\ell_a},\quad \eta=\frac{\ell_0}{\ell_a},\quad \phi_1(t)=\psi_1(s),\quad \phi_2(t)=\psi_2(s)$$
$$k_{ij}(x_1,t)=L_{ij}(r,s),\quad f_i(x_1)=g_i(r), \tag{30}$$

equations (27) and (28) may be expressed as

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\psi_i(s)}{s-r}\mathrm{d}s+\frac{1}{\pi}\int_{-1}^{1}\ell_a\sum_1^2 L_{ij}(r,s)\psi_j(s)\mathrm{d}s=g_i(r), \tag{31}$$
$$(i=1,2;|r|<1)$$

$$\int_{-1}^{+1}\psi_1(s)\mathrm{d}s=0,\qquad \int_{-1}^{+1}\psi_2(s)\mathrm{d}s=0. \tag{32}$$

Noting from (17) and (18) that the loading $g_i(r)$ of (31) presents jump discontinuities. Then, and following (Ioakimidis, 1980), $\psi_1(s)$ and $\psi_2(s)$ are replaced by new functions $\lambda_1(s)$ and $\lambda_2(s)$ so that

$$\psi_i(s)=\lambda_i(s)+h_i(s),\quad (i=1,2) \tag{33}$$

where $h_i(s)$ are the solutions of the following singular integral equations

$$\frac{1}{\pi}\int_{-1}^{+1}\frac{h_i(s)}{s-r}\mathrm{d}s=g_i(r),\quad (i=1,2;|r|<1), \tag{34}$$

supplemented by the conditions

$$\int_{-1}^{+1}h_i(s)\mathrm{d}s=0\quad (i=1,2). \tag{35}$$

Then the new unknown functions $\lambda_i(s)$ should satisfy the following system of singular integral equations

$$\frac{1}{\pi}\int_{-1}^{+1}\left(\frac{1}{s-r}+\ell_a\sum_1^2 L_{ij}(r,s)\right)\lambda_i(s)\mathrm{d}s=m_i(r),\quad (i=1,2,|r|\le 1). \tag{36}$$

with the conditions

$$\int_{-1}^{+1}\lambda_i(s)\mathrm{d}s=0, \tag{37}$$

where

$$m_i(r) = -\frac{\ell_a}{\pi}\left(\int_{-1}^{+1}\sum_{1}^{2} L_{ij}(r,s)h_j(s)\right)\mathrm{d}s \quad (i=1,2). \tag{38}$$

It is clear from equation (38) that, since $L_{ij}(r,s)$ are well-behaved, the same will hold true for $m_i(r)$ too and the numerical techniques for the solution of singular integral equations (Erdogan et al., 1973) can be directly applied to the solution of eqns (36) and (37) without any modifications. The solutions $\psi_i(s)$ of eqns (31) and (32) are given by equation (33).

The closed-form solutions of eqns (34) and (35), are determined by Gakhov (1966)

$$h_i(s) = -\frac{1}{\pi}\left(1-s^2\right)^{-\frac{1}{2}}\int_{-1}^{+1}\left(1-r^2\right)^{\frac{1}{2}}\frac{g_i(r)}{r-s}\mathrm{d}r \quad (i=1,2; |s|\le 1). \tag{39}$$

Next, performing the integration in (39) in closed form, yields

$$h_1(s) = -\frac{\kappa+1}{2\mu\pi}\tau_c\left(-2\sqrt{\frac{1-\eta^2}{1-s^2}}+\ln\left|\frac{\sqrt{1-\eta^2}+\sqrt{1-s^2}}{\sqrt{1-\eta^2}-\sqrt{1-s^2}}\right|\right),$$

$$h_2(s) = \frac{\kappa+1}{2\mu\pi}\quad\frac{s(-p_0\pi+2\sigma_c\arccos(\eta))}{\sqrt{1-s^2}} +\sigma_c\ln\left|\frac{\eta\sqrt{1-s^2}-s\sqrt{1-\eta^2}}{\eta\sqrt{1-s^2}+s\sqrt{1-\eta^2}}\right|\Bigg). \tag{40}$$

It was shown in Erdogan et al. (1973) that the system of singular integral equation (36) has an index 1 because the unknown functions $\lambda_i(s)$ have integrable singularities at the end points ±1. Its solution may be expressed as $\lambda_i(s) = w(s)\gamma_i(s)$ where $w(s)=(1-s^2)^{-1/2}$ is the weight function associated with the Chebyshev polynomials of the first kind $T_n(s)=\cos(n\arccos(s))$ and $\gamma_i(s)$ are continuous and bounded functions in the interval $[-1,1]$ which may be expressed as truncated series of Chebyshev polynomials of the first kind. Then, the solutions of (36) may be expressed as:

$$\lambda_1(s) = \left(1-s^2\right)^{-\frac{1}{2}}\sum_{n=0}^{N}A_nT_n(s), \quad \lambda_2(s) = \left(1-s^2\right)^{-\frac{1}{2}}\sum_{n=0}^{N}B_nT_n(s). \tag{41}$$

Injecting (41) into conditions (37), gives $A_0 = B_0 = 0$.

Substituting (41) into (36) and using the following relation

$$\frac{1}{\pi}\int_{-1}^{+1}\frac{T_n(s)\left(1-s^2\right)^{-\frac{1}{2}}}{s-r}\mathrm{d}s = \begin{cases} U_{n-1}(r), & n>0 \\ 0, & n=0\end{cases}. \tag{42}$$

where $U_n(r)=\sin((n+1)\arccos(r))/\sqrt{1-r^2}$ designates the Chebyshev polynomials of the second kind, results in

$$\begin{aligned}\sum_{n=1}^{N}\left(A_nU_{n-1}(r)+A_nH_n^{11}(r)+B_nH_n^{12}(r)\right) &= m_1(r),\\ \sum_{n=1}^{N}\left(B_nU_{n-1}(r)+A_nH_n^{21}(r)+B_nH_n^{22}(r)\right) &= m_2(r) \quad |r|<1,\end{aligned} \tag{43}$$

where

$$H_n^{ij}(r) = \frac{\ell_a}{\pi}\int_{-1}^{1}\left(1-s^2\right)^{-\frac{1}{2}}L_{ij}(r,s)T_n(s)\mathrm{d}s \quad (i,j=1,2). \tag{44}$$

Remark 3. *The integrals in (42) and (39) are improper integrals. They are evaluated in the Cauchy principal value sense.*

Equation (43) may be solved by selecting a set of N collocation points, as follows

$$r_j = \cos\left(\frac{(2j-1)\pi}{2N}\right), \quad j=1,..,N. \tag{45}$$

Using the collocation points given by equation (45) into equation (43) yields a system of $2N$ equations with $2N$ unknowns, namely $A_1,\ldots,A_N$ and $B_1,\ldots,B_N$ which may be expressed as:

$$\begin{aligned}\sum_{n=1}^{N}\left(A_n\left(U_{n-1}(r_j)+H_n^{11}(r_j)\right)+B_nH_n^{12}(r_j)\right) &= m_1(r_j)\\ \sum_{n=1}^{N}\left(B_n\left(U_{n-1}(r_j)+H_n^{22}(r_j)\right)+A_nH_n^{21}(r_j)\right) &= m_2(r_j) \quad j=1,\ldots,N.\end{aligned} \tag{46}$$

The stress intensity factors at the crack tip $(\ell_a,0)$ is given by

$$\begin{aligned}k_1(\ell_a) &= -\frac{2\mu}{(\kappa+1)\sqrt{\ell_a}}\lim_{x_1\to\ell_a}\sqrt{\ell_a^2-x_1^2}\phi_2(x_1) = -\frac{2\mu\sqrt{\ell_a}}{(\kappa+1)}\lim_{s\to 1}\sqrt{1-s^2}\psi_2(s)\\ k_2(\ell_a) &= -\frac{2\mu}{(\kappa+1)\sqrt{\ell_a}}\lim_{x_1\to\ell_a}\sqrt{\ell_a^2-x_1^2}\phi_1(x_1) = -\frac{2\mu\sqrt{\ell_a}}{(\kappa+1)}\lim_{s\to 1}\sqrt{1-s^2}\psi_1(s)\end{aligned} \tag{47}$$

Substituting from (33), (40) and (41) into (47) gives:

$$\begin{aligned}k_1(\ell_a) &= -\sqrt{\ell_a}\quad -p_0+\frac{2}{\pi}\sigma_c\arccos(\eta)+\frac{2\mu}{\kappa+1}\sum_{1}^{N}B_n\Bigg)\\ k_2(\ell_a) &= -2\sqrt{\ell_a}\quad\frac{\tau_c}{\pi}\sqrt{1-\eta^2}+\frac{\mu}{\kappa+1}\sum_{1}^{N}A_n\Bigg).\end{aligned} \tag{48}$$

Remark 4. *For the left crack tip we have* $k_1(-\ell_a) = -k_1(\ell_a)$ *and* $k_2(-\ell_a) = k_2(\ell_a)$.

The crack gaps at $x_1 \in [-\ell_a, \ell_a]$ are defined by:

$$
\begin{aligned}
[\![u_1]\!](x_1) &= u_1\left(x_1, 0^+\right) - u_1\left(x_1, 0^-\right) = \int_{-\ell_a}^{x_1} \phi_1(t)\mathrm{d}t \\
[\![u_2]\!](x_1) &= u_2\left(x_1, 0^+\right) - u_2\left(x_1, 0^-\right) = \int_{-\ell_a}^{x_1} \phi_2(t)\mathrm{d}t.
\end{aligned}
\tag{49}
$$

Substituting from (33), (40) and (41) into (49) results in:

$$
[\![u_1]\!](x_1) = -\frac{(1+\kappa)\tau_c}{2\pi\mu}\left(x_1 \ln\left|\frac{\sqrt{\ell_a^2-\ell_0^2}+\sqrt{\ell_a^2-x_1^2}}{\sqrt{\ell_a^2-\ell_0^2}-\sqrt{\ell_a^2-x_1^2}}\right| + \ell_0 \ln\left|\frac{\left(\ell_a^2+x_1\ell_0+\sqrt{(\ell_a^2-\ell_0^2)(\ell_a^2-x_1^2)}\right)(x_1-\ell_0)}{\left(\ell_a^2-x_1\ell_0+\sqrt{(\ell_a^2-\ell_0^2)(\ell_a^2-x_1^2)}\right)(x_1+\ell_0)}\right|\right) - \sqrt{\ell_a^2-x_1^2}\sum_1^N A_n \frac{U_{n-1}(x_1/\ell_a)}{n}
$$

$$
\begin{aligned}
[\![u_2]\!](x_1) = \frac{(1+\kappa)}{2\pi\mu}&\left(\pi p_0\sqrt{\ell_a^2-x_1^2} + \sigma_c\left(x_1 \ln\left|\frac{x_1\sqrt{\ell_a^2-\ell_0^2}-\ell_0\sqrt{\ell_a^2-x_1^2}}{x_1\sqrt{\ell_a^2-\ell_0^2}+\ell_0\sqrt{\ell_a^2-x_1^2}}\right| + \ell_0\ln\left|\frac{\left(\sqrt{\ell_a^2-x_1^2}+\sqrt{\ell_a^2-\ell_0^2}\right)^2}{x_1^2-\ell_0^2}\right| - 2\arccos(\eta)\sqrt{\ell_a^2-x_1^2}\right)\right) \\
&-\sqrt{\ell_a^2-x_1^2}\sum_1^N B_n\frac{U_{n-1}(x_1/\ell_a)}{n}.
\end{aligned}
\tag{50}
$$

It may be observed from (50) that the gaps present logarithmic singularities at $x_1 = \pm\ell_0$. So the gaps at these points are defined as the limits of (50) when $x_1 \to \pm\ell_0$:

$$
\begin{aligned}
\delta_1(\ell_0) &= \lim_{x_1\to\ell_0}[\![u_1(x_1)]\!] = \frac{1+\kappa}{\mu\pi}\tau_c\ell_0\ln(\eta) - \sqrt{\ell_a^2-\ell_0^2}\sum_1^N\frac{A_nU_{n-1}(\eta)}{n} \\
\delta_2(\ell_0) &= \lim_{x_1\to\ell_0}[\![u_2(x_1)]\!] = \frac{1+\kappa}{2\mu}\left(\frac{-2\sigma_c\ell_0\ln(\eta)}{\pi} + \sqrt{\ell_a^2-\ell_0^2}\left(p_0 - \frac{2\sigma_c\arccos(\eta)}{\pi}\right)\right) - \sqrt{\ell_a^2-\ell_0^2}\sum_1^N\frac{B_nU_{n-1}(\eta)}{n}.
\end{aligned}
\tag{51}
$$

Remark 5. *From the symmetry of the problem, it is clear that* $\delta_1(-\ell_0) = -\delta_1(\ell_0)$ *and* $\delta_2(-\ell_0) = \delta_2(\ell_0)$.

Remark 6. *In the case where* $h \to \infty$, *the kernels* $k_{ij}(x_1,t) = 0$ *(see (29)). Consequently* $m_i(r) = 0$, $\lambda_i(s) = 0$ *and* $A_n = B_n = 0$ *(see (36), (38) and (41)). The condition* $k_2 = 0$ *induces* $\tau_c = 0$, *and* $k_1 = 0$ *induces* $p_0 = 2/\pi\sigma_c\arccos(\eta)$ *(see (48)). The mode I solution of a crack in infinite medium is recovered.*

5. Numerical procedure and results

The problem can be described with the set of the following dimensionless parameters $A = 2\mu/(\kappa+1)\sigma_c$, $B = G_c/\sigma_c\ell_0$ and h/ℓ_0. In this study, the values of G_c, σ_c, ℓ_0, μ and κ are set such that:

$$
A = 10, \quad B = 1e-1
$$

The variable parameter is the strip height h.

Remark 7. *In the Dugdale model (paragraph 2.2), there are three parameters:* σ_c, τ_c *and* G_c. *It is thus assumed* σ_c *and* G_c *material constants.*

As seen in paragraph 3.1, there are two phases in the evolution of the crack: the cohesive phase and the propagation phase. Presented below is the numerical method used in each phase.

5.1. Numerical procedure in the cohesive phase

The criterion governing the evolution of the process zone tips $(\pm\ell_a, 0)$ is $k_1(\pm\ell_a) = k_2(\pm\ell_a) = 0$. These are implicit equations linking p_0, τ_c and ℓ_a. From a practical viewpoint, it is easier to calculate p_0 and τ_c by supposing known ℓ_a. Indeed, by using the linearity of the elastic problem the stress intensity factors $k_1(\ell_a)$ and $k_2(\ell_a)$ can read as

$$
\begin{aligned}
k_1(\ell_a) &= p_0k_1^0(\ell_a) + \sigma_ck_1^{\sigma_c}(\ell_a) + \tau_ck_1^{\tau_c}(\ell_a), \\
k_2(\ell_a) &= p_0k_2^0(\ell_a) + \sigma_ck_2^{\sigma_c}(\ell_a) + \tau_ck_2^{\tau_c}(\ell_a),
\end{aligned}
\tag{52}
$$

where the different stress intensity factors in (52) are computed for different cases of loadings:

- $k_1^0(\ell_a)$ and $k_2^0(\ell_a)$ for $p_0 = 1$, $\sigma_c = \tau_c = 0$
- $k_1^{\sigma_c}(\ell_a)$ and $k_2^{\sigma_c}(\ell_a)$ for $\sigma_c = 1$, $p_0 = \tau_c = 0$
- $k_1^{\tau_c}(\ell_a)$ and $k_2^{\tau_c}(\ell_a)$ for $\tau_c = 1$, $\sigma_c = p_0 = 0$.

Equations $k_1(\ell_a) = k_2(\ell_a) = 0$ lead to:

$$
\begin{aligned}
p_0 &= \frac{\sigma_c\left(-k_1^{\sigma_c}(\ell_a)k_2^{\tau_c}(\ell_a) + k_2^{\sigma_c}(\ell_a)k_1^{\tau_c}(\ell_a)\right)}{k_1^0(\ell_a)k_2^{\tau_c}(\ell_a) - k_2^0(\ell_a)k_1^{\tau_c}(\ell_a)} \\
\tau_c &= \frac{\sigma_c\left(-k_1^{\sigma_c}(\ell_a)k_2^0(\ell_a) + k_2^{\sigma_c}(\ell_a)k_1^0(\ell_a)\right)}{k_2^0(\ell_a)k_1^{\tau_c}(\ell_a) - k_2^{\tau_c}(\ell_a)k_1^0(\ell_a)}.
\end{aligned}
\tag{53}
$$

Specifically, for a given value of ℓ_a, the load p_0 and the cohesive force τ_c are determined by formulas (53). This requires to compute the different stress intensity factors by formulas (48).

Remark 8. *It is observed from (53) that the critical stress τ_c (or the ratio τ_c/σ_c) is not a material constant. Indeed it is related to ℓ_a, or to the length of the process zone (for a given h). The fact that the ratio τ_c/σ_c is not a material constant has already been observed in Becker and Gross (1988) in the case of an infinite medium.*

5.2. Numerical procedure in the propagation phase

In this phase, the value of the traction free crack tip ℓ_c is prescribed and the values of the load p_0, the critical stress τ_c and of the process zone tip ℓ_a computed by solving the system of non linear equation (25). The numerical method used is the following: For a given entry value of ℓ_a, p_0 and τ_c are obtained by solving the equations $k_1 = k_2 = 0$ with the procedure explained in paragraph 5.1. The right value of ℓ_a is obtained by dichotomy so that $\sigma_c[\![u_2]\!](\pm\ell_c) + \tau_c|[\![u_1]\!](\pm\ell_c)| = G_c$.

5.3. Results

It can be seen in Fig. 2, the evolution of the applied load p_0 with the position of the process zone tip ℓ_a, for different values of h. Two different parts on the curves may be observed: an increasing part, corresponding to the cohesive phase, and a decreasing part, corresponding to the propagation phase. It can also be observed that the larger the height h, the more important the applied load (for a given ℓ_a).

The evolution of the critical tangential stress τ_c with the position of the process zone tip ℓ_a, for different values of h, is presented in Fig. 3. As in the previous case, there are two different parts on the curves. These parts correspond to the cohesive and propagation phases. Finally, it may be noted that the larger the height, the weaker τ_c and the problem approaches the mode I case.

In Fig. 4 the shape of the normal gap $[\![u_2]\!]$ between the crack faces is presented, for different values of h and the same value of $\ell_a/\ell_0 = 2$. It is observed that the larger the height, the weaker the normal gap. This latter tends to the infinite medium case gap. Also, the curves present an inflection point at $x_1 = \ell_0$. This point corresponds to the beginning of the process zone.

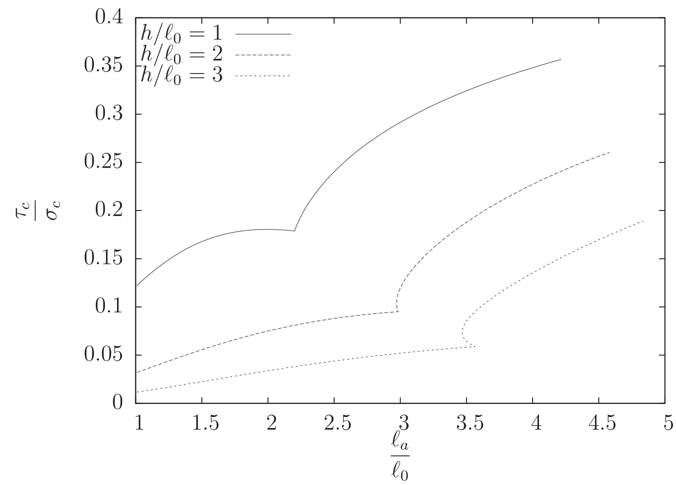

Fig. 3. Evolution of τ_c/σ_c with the relative crack length ℓ_a/ℓ_0 for different values of h/ℓ_0.

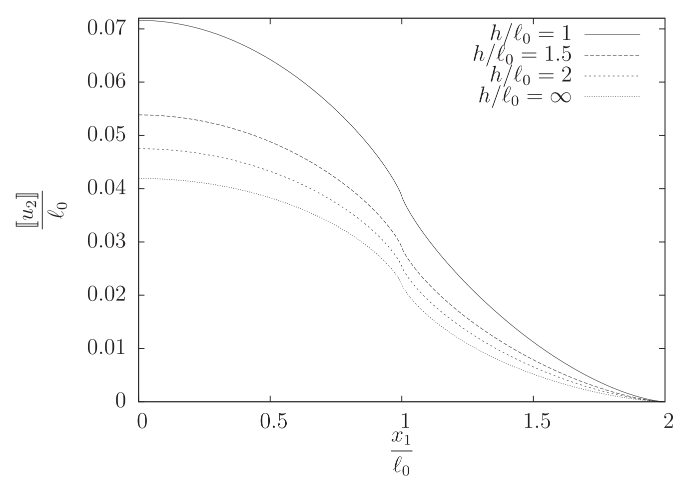

Fig. 4. Relative normal crack gaps for different values of h/ℓ_0 and the same value of $\ell_a/\ell_0 = 2$.

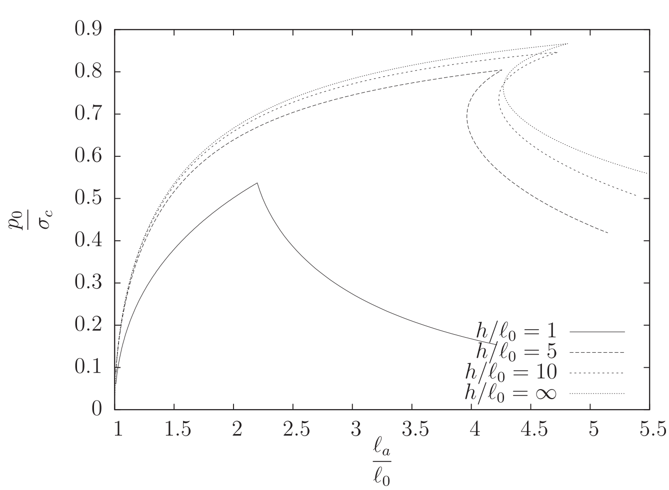

Fig. 2. Evolution of p_0/σ_c with the relative crack length ℓ_a/ℓ_0 for different values of h/ℓ_0.

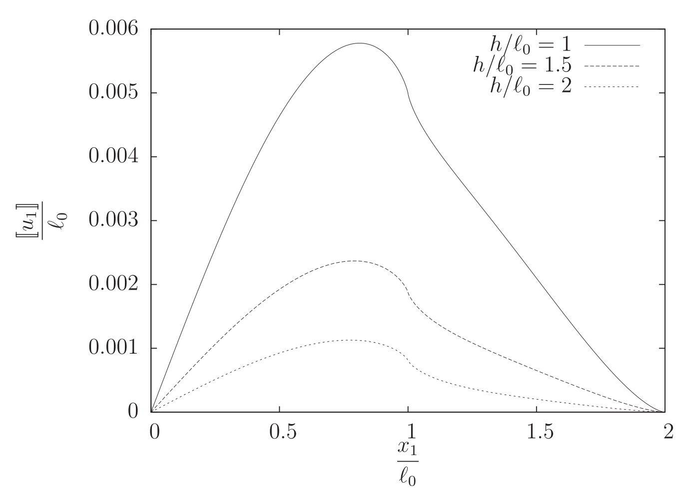

Fig. 5. Relative tangential crack gaps for different values of h/ℓ_0 and the same value of $\ell_a/\ell_0 = 2$.

The shape of the tangential gap $[\![u_1]\!]$ between the crack faces is presented in Fig. 5, for different values of h and the same value of $\ell_a/\ell_0 = 2$. It is noted that for all the curves the gap is nil at $x_1 = 0$. This is due to the symmetry of the problem. Also, the larger the height, the weaker the tangential gap. This latter tends to the infinite medium case gap. The curves present an inflection point at $x_1 = \ell_0$. This point corresponds to the beginning of the process zone.

6. Conclusion

The most important results of this work are the following:

- The semi-analytical solution of a semi-infinite structure with a Dugdale crack under a mixed mode loading has been established.
- The general cohesive zone model in the mixed mode case was established via a variational formulation based on the revisited Griffith theory, and the particular case of the Dugdale model was deduced.
- The system of singular integral equations of the first kind was solved for the Dugdale case where the loading is discontinuous on the crack faces. The resolution method used takes into account this discontinuity.
- The ratio of critical stresses σ_c/τ_c is not a material constant, but depends on the length of the process zone and the height h of the strip. The fact that σ_c/τ_c is not a material constant is in agreement with the results obtained by Becker and Gross (1988) in the case of a crack in an infinite medium.

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