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► **To cite this version:**

Emmanuel Gobet, Gang Liu, Jorge Zubelli. A NON-INTRUSIVE STRATIFIED RESAMPLER FOR REGRESSION MONTE CARLO: APPLICATION TO SOLVING NON-LINEAR EQUATIONS . 2016. <hal-01291056>

HAL Id: hal-01291056

<https://hal-polytechnique.archives-ouvertes.fr/hal-01291056>

Submitted on 20 Mar 2016

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1 **A NON-INTRUSIVE STRATIFIED RESAMPLER FOR REGRESSION**
2 **MONTE CARLO: APPLICATION TO SOLVING NON-LINEAR**
3 **EQUATIONS***

4 EMMANUEL GOBET[†], GANG LIU[†], AND JORGE P. ZUBELLI[‡]

5 **Abstract.** Our goal is to solve certain dynamic programming equations associated to a given
6 Markov chain X , using a regression-based Monte Carlo algorithm. More specifically, we assume that
7 the model for X is not known in full detail and only a root sample X^1, \dots, X^M of such process
8 is available. By a stratification of the space and a suitable choice of a probability measure ν , we
9 design a new resampling scheme that allows to compute local regressions (on basis functions) in each
10 stratum. The combination of the stratification and the resampling allows to compute the solution to
11 the dynamic programming equation (possibly in large dimensions) using only a relatively small set
12 of root paths. To assess the accuracy of the algorithm, we establish non-asymptotic error estimates
13 in $L^2(\nu)$. Our numerical experiments illustrate the good performance, even with $M = 20 - 40$ root
14 paths.

15 **Key words.** discrete Dynamic Programming Equations, empirical regression scheme, resam-
16 pling methods, small-size sample

17 **AMS subject classifications.** 62G08, 62G09, 93Exx

18 **1. Introduction.** Stochastic dynamic programming equations are classic equa-
19 tions arising in the resolution of nonlinear evolution equations, like in stochastic control
20 (see [18, 4]) or non-linear PDEs (see [6, 9]). In a discrete-time setting they take
21 the form:

22
$$Y_N = g_N(X_N), \quad Y_i = \mathbb{E} [g_i(Y_{i+1}, \dots, Y_N, X_i, \dots, X_N) \mid X_i], \quad i = N - 1, \dots, 0,$$

24 for some functions g_N and g_i which depend on the non-linear problem under consid-
25 eration. Here $X = (X_0, \dots, X_N)$ is a Markov chain valued in \mathbb{R}^d , entering also in
26 the definition of the problem. The aim is to compute the value function y_i such that
27 $Y_i = y_i(X_i)$.

28 Among the popular methods to solve this kind of problem, we are concerned with
29 Regression Monte Carlo (RMC) methods that take as input M simulated paths of X ,
30 say $(X^1, \dots, X^M) =: X^{1:M}$, and provide as output simulation-based approximations
31 $y_i^{M,\mathcal{L}}$ using Ordinary Least Squares (OLS) within a vector space of functions \mathcal{L} :

32
$$y_i^{M,\mathcal{L}} = \arg \inf_{\varphi \in \mathcal{L}} \frac{1}{M} \sum_{m=1}^M \left| g_i(y_{i+1}^{M,\mathcal{L}}(X_{i+1}^m), \dots, y_N^{M,\mathcal{L}}(X_N^m), X_i^m, \dots, X_N^m) - \varphi(X_i^m) \right|^2.$$

34 This Regression Monte Carlo methodology has been investigated in [9] to solve Back-
35 ward Stochastic Differential Equations associated to semi-linear partial differential
36 equations (PDEs) [16], with some tight error estimates. Generally speaking, it is well
37 known that the number of simulations M has to be much larger than the dimension
38 of the vector space \mathcal{L} and thus the number of coefficients we are seeking.

*This work is part of the Chair *Financial Risks* of the *Risk Foundation*, the *Finance for Energy Market Research Centre* and the ANR project *CAESARS* (ANR-15-CE05-0024).

[†]Centre de Mathématiques Appliquées (CMAP), Ecole Polytechnique and CNRS, Université Paris-Saclay, Route de Saclay, 91128 Palaiseau Cedex, France (Email: emmanuel.gobet@polytechnique.edu (Corresponding author), gang.liu1988@gmail.com).

[‡]IMPA, Est. D. Castorina 110, Rio de Janeiro, RJ 22460-320, Brazil (Email: zubelli@impa.br).

39 In contradistinction, throughout this work, we focus on the case where M is
 40 relatively small (a few hundreds) and the simulations are not sampled by the user but
 41 are directly taken from historical data ($X^{1:M}$ is called **root sample**), in the spirit of
 42 [17]. This is the most realistic situation when we collect data and when the model
 43 which fits the data is unknown.

44 Thus, as a main difference with the aforementioned references:

- 45 • We do not assume that we have full information about the model for X and
 46 we do not assume that we can generate as many simulations as needed to
 47 have convergent Regression Monte Carlo methods.
- 48 • The size M of the learning samples X^1, \dots, X^M is relatively small, which
 49 discards the use of a direct RMC with large dimensional \mathcal{L} .

50 To overcome these major obstacles, we elaborate on two ingredients:

- 51 1. First, we partition \mathbb{R}^d in strata $(\mathcal{H}_k)_k$, so that the regression functions can
 52 be computed locally on each stratum \mathcal{H}_k ; for *small* stratum this allows to
 53 use only a small dimensional approximation space \mathcal{L}_k , and therefore it puts
 54 a lower constraint on M . In general, this stratification breaks the properties
 55 for having well-behaved error propagation and we provide a precise way to
 56 sample in order to be able to aggregate the error estimates in different strata.
 57 We use a probabilistic distribution ν that has good norm-stability properties
 58 with X (see Assumptions 3.2 and 4.2).
- 59 2. Second, by assuming a mild model condition on X , we are able to resample
 60 from the root sample of size M , a *training sample* of M simulations suitable
 61 for the stratum \mathcal{H}_k . This resampler is non intrusive in the sense that it only
 62 requires to know the form of the model but not its coefficients: for example,
 63 we can handle models with independent increments (discrete inhomogeneous
 64 Levy process) or Ornstein-Uhlenbeck processes. See Examples 2.1-2.2-2.3-
 65 2.4. We call this scheme NISR (Non Intrusive Stratified Resampler), it is
 66 described in Definition 2.1 and Proposition 2.1.

67 The resulting regression scheme is, to the best of our knowledge, completely new.

68 To sum up, the contributions of this work are the following:

- 69 • We design a non-intrusive stratified resample (NISR) scheme that allows to
 70 sample from M paths of the root sample restarting from any stratum \mathcal{H}_k .
 71 See Section 2.
- 72 • We combine this with regression Monte Carlo schemes, in order to solve one-
 73 step ahead dynamic programming equations (Section 3), discrete backward
 74 stochastic differential equations (BSDEs) and semi-linear PDEs (Section 4).
- 75 • In Theorems 3.4 and 4.1, we provide quadratic error estimates of the form

$$76 \quad \text{quadratic error on } y_i \leq \text{approximation error} + \text{statistical error} \\
 77 \quad \quad \quad + \text{interdependency error} .$$

79 The approximation error is related to the best approximation of y_i on each
 80 stratum \mathcal{H}_k , and averaged over all the strata. The statistical error is bounded
 81 by C/M with a constant C which does not depend on the number of strata:
 82 only relatively small M is necessary to get low statistical errors. This is in
 83 agreement with the motivation that the root sample has a relatively small
 84 size. The interdependency error is an unusual issue, it is related to the strong
 85 dependency between regression problems (because they all use the same root
 86 sample). The analysis as well as the framework are original. The error es-
 87 timates take different forms according to the problem at hand (Section 3 or

Section 4).

- Finally we illustrate the performance of the methods on two types of examples: first, approximation of non-linear PDEs arising in reaction-diffusion biological models (Subsection 5.1) and optimal sequential decision (Subsection 5.2), where we illustrate that root samples of size $M = 20 - 40$ only can lead to remarkably accurate numerical solutions.

The paper is organized as follows. In Section 2 we present the model structure that leads to the non-intrusive stratified resampler for regression Monte Carlo (NISR), together with the stratification. Main notations will be also introduced. The algorithm is presented in a generic form of dynamic programming equations in Algorithm 1. In Section 3 we analyze the convergence of the algorithm in the case of one-step ahead dynamic programming equations (for instance optimal stopping problems). Section 4 is devoted to the convergence analysis for discrete BSDEs (probabilistic representation of semi-linear PDEs arising in stochastic control problems). Section 5 is devoted to numerical examples. Technical results are postponed to the Appendix.

2. Setting and the general algorithm.

2.1. General dynamic programming equation. Suppose we have N discrete dates, and we aim at solving numerically the following dynamic programming equation (DPE for short), written in general form:

$$Y_N = g_N(X_N), \quad Y_i = \mathbb{E}[g_i(Y_{i+1:N}, X_{i:N}) \mid X_i], \quad 0 \leq i < N.$$

Here, $(X_i)_{0 \leq i \leq N}$ is a Markov chain with state space \mathbb{R}^d , $(Y_i)_{0 \leq i \leq N}$ is a random process taking values in \mathbb{R} and we use for convenience the generic short notation $z_{i:N} := (z_i, \dots, z_N)$. Note that the argument of the conditional expectation is path-dependent, thus allowing greater generality. Had we considered Y to be multidimensional, the subsequent algorithm and analysis would remain essentially the same.

Later (Sections 3 and 4), specific forms for g_i will be considered, depending on the model of DPE to solve at hand: it will have an impact on the error estimates that we can derive. However, the description of the algorithm can be the same for all the DPEs, as seen below, and this justifies our choice of unifying the presentation.

Thanks to the Markovian property of X , under mild assumptions we can easily prove by induction that there exists a measurable function y_i such that $Y_i = y_i(X_i)$, our aim is to compute an approximation of the value functions $y_i(\cdot)$ for all i . We assume that a bound on y_i is available.

ASSUMPTION 2.1 (A priori bound). *The solution y_i is bounded by a constant $|y_i|_\infty$.*

2.2. Model structure and root sample. We will represent $y_i(\cdot)$ through its coefficients on a vector space, and the coefficients will be computed thanks to learning samples of X .

ASSUMPTION 2.2 (Data). *We have the observation of M independent paths of X , which are denoted by $((X_i^m : 0 \leq i \leq N), 1 \leq m \leq M)$. We refer to this data as the root sample.*

For our needs, we adopt a representation of the *flow of the Markov chain for different initial conditions*, i.e., the Markov chain $X^{i,x}$ starting at different times $i \in \{0, \dots, N\}$ and points $x \in \mathbb{R}^d$. Namely, we write

$$(2.1) \quad X_j^{i,x} = \theta_{i,j}(x, U), \quad i \leq j \leq N,$$

where

- 134 • U is some random vector, called random source,
 135 • $\theta_{i,j}$ are (deterministic) measurable functions.

136 We emphasize that, for the sake of convenience, U is the same for representing all
 137 $X_j^{i,x}, 0 \leq i \leq j \leq N, x \in \mathbb{R}^d$.

ASSUMPTION 2.3 (Noise extraction). *We assume that $\theta_{i,j}$ are known and we can retrieve the random sources (U^1, \dots, U^M) associated to the root sample $X^{1:M} = (X^m : 1 \leq m \leq M)$, i.e.,*

$$X_j^m = X_j^{0,x_0^m,m} = \theta_{0,j}(x_0^m, U^m).$$

138 Observe that this assumption is much less stringent than identifying the distribution
 139 of the model. We exemplify this now.

EXAMPLE 2.1 (Arithmetic Brownian motion with time dependent parameters). *Let $(t_i : 0 \leq i \leq N)$ be N times and define the arithmetic Brownian motion by*

$$X_i = x_0 + \int_0^{t_i} \mu_s ds + \int_0^{t_i} \sigma_s dW_s$$

where $\mu_t \in \mathbb{R}^d, \sigma_t \in \mathbb{R}^{d \times q}, W_t \in \mathbb{R}^q$ and μ, σ are deterministic functions of time. In this case, the random source is given by

$$U := (X_{i+1} - X_i)_{0 \leq i \leq N-1}$$

and the functions by

$$\theta_{ij}(x, U) := x + \sum_{i \leq k < j} U_k.$$

140 This works since $U_i = \int_{t_i}^{t_{i+1}} \mu_s ds + \int_{t_i}^{t_{i+1}} \sigma_s dW_s$. The crucial point is that, in order
 141 to extract U from X , we do not assume that μ and σ are known.

EXAMPLE 2.2 (Levy process). *More generally, we can set $X_i = \mathbf{X}_{t_i}$ with a time-inhomogeneous Levy process \mathbf{X} . Then take*

$$U := (X_{i+1} - X_i)_{0 \leq i \leq N-1}, \quad \theta_{ij}(x, U) := x + \sum_{i \leq k < j} U_k.$$

EXAMPLE 2.3 (Geometric Brownian motion with time dependent parameters). *With the same kind of parameters as for Example 2.1, define the geometric Brownian motion (component by component)*

$$X_i = X_0 \exp \left(\int_0^{t_i} \mu_s ds + \int_0^{t_i} \sigma_s dW_s \right).$$

Then, we have that

$$U := \left(\log \left(\frac{X_{i+1}}{X_i} \right) \right)_{0 \leq i \leq N-1}, \quad \theta_{ij}(x, U) := x \prod_{i \leq k < j} \exp(U_k).$$

EXAMPLE 2.4 (Ornstein-Uhlenbeck process with time dependent parameters). *Given N times $(t_i : 0 \leq i \leq N)$, set $X_i = \mathbf{X}_{t_i}$ where \mathbf{X} has the following dynamics:*

$$\mathbf{X}_t = \mathbf{x}_0 - \int_0^t A(\mathbf{X}_s - \bar{\mathbf{X}}_s) ds + \int_0^t \Sigma_s dW_s$$

where A is $d \times d$ -matrix, \mathbf{X}_t and $\bar{\mathbf{X}}_t$ are in \mathbb{R}^d , Σ_t is a $d \times q$ -matrix, $W_t \in \mathbb{R}^q$. $\bar{\mathbf{X}}_t$ and Σ_t are both deterministic functions of time. The explicit solution is

$$\mathbf{X}_t = e^{-A(t-s)}\mathbf{X}_s + e^{-At} \int_s^t e^{Ar} (A\bar{\mathbf{X}}_r dr + \Sigma_r dW_r).$$

Assume that we know A : in this case, an observation of $X_{0:N}$ enables to retrieve the random source

$$U := \left(X_j - e^{-A(t_j-t_i)} X_i \right)_{0 \leq i \leq j \leq N}$$

and then

$$\theta_{ij}(x, U) := e^{-A(t_j-t_i)} x + U_{i,j}.$$

142 The noise extraction works since $U_{i,j} = e^{-At_j} \int_{t_i}^{t_j} e^{Ar} (A\bar{\mathbf{X}}_r dr + \Sigma_r dW_r)$.

143 As illustrated above, through Assumption 2.2, all we need to know is the general
 144 structure of the Markov chain model but we do not need to estimate all the model
 145 parameters, and sometimes none of them (Examples 2.1, 2.2, 2.3). Our approach is
 146 non intrusive in this sense.

147 **2.3. Stratification and resampling algorithm.** On the one hand, we can rely
 148 on a root sample of size M only (possibly with a relatively small M , constrained by
 149 the available data), which is very little to perform accurate Regression Monte-Carlo
 150 methods (usually M has to be much larger than the dimension of approximation
 151 spaces, as reminded in introduction).

152 On the other hand, we are able to access the random sources so that resampling
 153 the M paths is possible. The degree of freedom comes from the flexibility of initial
 154 conditions (i, x) , thanks to the flow representation (2.1). We now explain how we take
 155 advantage of this property.

156 The idea is to resample the model paths for different starting points in different
 157 parts of the space \mathbb{R}^d and on each part, we will perform a regression Monte Carlo
 158 using M paths and a low-dimensional approximation space. These ingredients give
 159 the ground reasons for getting accurate results.

160 Let us proceed to the details of the algorithm. We design a stratification approach:
 161 suppose there exist K strata $(\mathcal{H}_k)_{1 \leq k \leq K}$ such that

$$162 \quad \mathcal{H}_k \cap \mathcal{H}_l = \emptyset \quad \text{for } k \neq l, \quad \bigcup_{k=1}^K \mathcal{H}_k = \mathbb{R}^d.$$

163 An example for \mathcal{H}_k is a hypercube of the form $\mathcal{H}_k = \prod_{l=1}^d [x_{k,l}^-, x_{k,l}^+)$. Then, we are
 164 given a probability measure ν on \mathbb{R}^d and denote its restriction on \mathcal{H}_k by

$$165 \quad \nu_k(dx) := \frac{1}{\nu(\mathcal{H}_k)} 1_{\mathcal{H}_k}(x) \nu(dx).$$

166 The measure ν will serve as a reference to control the errors. See Paragraph 3.1.2 and
 167 Section 5 for choices of ν .

168 **DEFINITION 2.1** (Non-intrusive stratified resampler, NISR for short). *We define the*
 169 *M -sample used for regression at time i and in the k -th stratum \mathcal{H}_k :*

- 170 • let $(X_i^{i,k,m})_{1 \leq m \leq M}$ be an i.i.d. sample according to the law ν_k ;

- for $j = i + 1, \dots, N$, set

$$X_j^{i,k,m} = \theta_{i,j}(X_i^{i,k,m}, U^m),$$

171 where $U^{1:M}$ are the random sources from Assumption 2.3.

172 In view of Assumptions 2.2 and 2.3, the random sources U^1, \dots, U^M are independent, therefore we easily prove the following.

174 PROPOSITION 2.1. *The M paths $(X_{i:N}^{i,k,m}, 1 \leq m \leq M)$ are independent and identically distributed as $X_{i:N}$ with $X_i \stackrel{d}{\sim} \nu_k$.*

176 **2.4. Approximation spaces and regression Monte Carlo schemes.** On each stratum, we approximate the value functions y_i using basis functions. We can take different kinds of basis functions:

- 179 - **LP₀** (partitioning estimate): $\mathcal{L}_k = \text{span}(1_{\mathcal{H}_k})$,
- 180 - **LP₁** (piecewise linear): $\mathcal{L}_k = \text{span}(1_{\mathcal{H}_k}, x_1 1_{\mathcal{H}_k}, \dots, x_d 1_{\mathcal{H}_k})$,
- 181 - **LP_n** (piecewise polynomial): $\mathcal{L}_k = \text{span}$ (all the polynomials of degree less than or equal to n on \mathcal{H}_k).

To simplify the presentation, we assume hereafter that the dimension of \mathcal{L}_k does not depend on k , we write

$$\dim(\mathcal{L}_k) =: \dim(\mathcal{L}).$$

183 To compute the approximation of y_i on each stratum \mathcal{H}_k , we will use the M samples of Definition 2.1. Our NISR-regression Monte Carlo algorithm takes the form:

Algorithm 1 General NISR-regression Monte Carlo algorithm

```

1: set  $y_N^{(M)}(\cdot) = g_N(\cdot)$ 
2: for  $i = N - 1$  until 0 do
3:   for  $k = 1$  until  $K$  do
4:     sample  $(X_{i:N}^{i,k,m})_{1 \leq m \leq M}$  using the NISR (Definition 2.1)
5:     set  $S^{(M)}(x_{i:N}) = g_i(y_{i+1}^{(M)}(x_{i+1}), \dots, y_N^{(M)}(x_N), x_{i:N})$ 
6:     compute  $\psi_i^{(M),k} = \text{OLS}(S^{(M)}, \mathcal{L}_k, X_{i:N}^{i,k,1:M})$ 
7:     set  $y_i^{(M),k} = \mathcal{T}_{|y_i|_\infty}(\psi_i^{(M),k})$  where  $\mathcal{T}_L$  is the truncation operator,
8:     defined by  $\mathcal{T}_L(x) = -L \vee x \wedge L$ 
9:   end for
10:  set  $y_i^{(M)} = \sum_{k=1}^K y_i^{(M),k} 1_{\mathcal{H}_k}$ 
11: end for

```

In the above, the Ordinary Least Squares approximation of the response function $\tilde{S} : (\mathbb{R}^d)^{N-i+1} \mapsto \mathbb{R}$ in the function space \mathcal{L}_k using the M sample $X_{i:N}^{i,k,1:M}$ is defined and denoted by

$$\text{OLS}(\tilde{S}, \mathcal{L}_k, X_{i:N}^{i,k,1:M}) = \arg \inf_{\varphi \in \mathcal{L}_k} \frac{1}{M} \sum_{m=1}^M |\tilde{S}(X_{i:N}^{i,k,m}) - \varphi(X_i^{i,k,m})|^2.$$

185 The main difference with the usual regression Monte-Carlo schemes (see [8] for
186 instance) is that here we use the common random numbers $U^{1:M}$ for all the regression
187 problems. This is the effect of resampling. The convergence analysis becomes more
188 delicate because we lose nice independence properties. Figure 1 describes a key part
189 in the algorithm, namely the process of using the root paths to generate new paths.

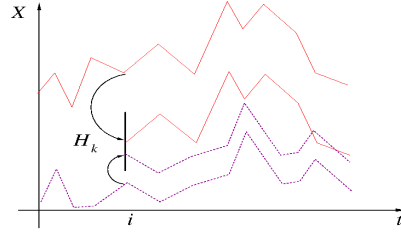


FIG. 1. Description of the use of the root paths to produce new paths in an arbitrary hypercube.

190 **3. Convergence analysis in the case of the one-step ahead dynamic**
 191 **programming equation.** We consider here the case

192
$$Y_N = g_N(X_N), \quad Y_i = \mathbb{E} [g_i(Y_{i+1}, X_i, \dots, X_N) \mid X_i], \quad 0 \leq i < N,$$

194 where we need the value of Y_{i+1} (one step ahead) to compute the value Y_i (at the
 195 current date) through a conditional expectation. To compare with Algorithm 1, we
 196 take $g_i(Y_{i+1:N}, X_{i:N}) = g_i(Y_{i+1}, X_{i:N})$.

Equations of this form are quite natural when solving optimal stopping problems
 in the Markovian case. Indeed, if V_i is the related value function at time i , i.e., the
 essential supremum over stopping times $\tau \in \{i, \dots, N\}$ of a reward process $f_\tau(X_\tau)$,
 then $V_i = \max(Y_i, f_i(X_i))$ where Y_i is the continuation value defined by

$$Y_i = \mathbb{E} [\max(Y_{i+1}, f_{i+1}(X_{i+1})) \mid X_i] ,$$

see [18] for instance. This corresponds to our setting with

$$g_i(y_{i+1}, x_{i:N}) = \max(y_{i+1}, f_{i+1}(x_{i+1})) .$$

197 Similar dynamic programming equations appear in stochastic control problems. See
 198 [4].

199 **3.1. Standing assumptions.** The following assumptions enable us to provide
 200 error estimates (Theorem 3.4 and Corollary 3.1) for the convergence of Algorithm 1.

201 **3.1.1. Assumptions on g_i .**

202 ASSUMPTION 3.1 (Functions g_i). *Each function g_i is Lipschitz w.r.t. the variable*
 203 *y_{i+1} , with Lipschitz constant L_{g_i} and $C_{g_i} := \sup_{x_{i:N}} |g_i(0, x_{i:N})| < +\infty$.*

204 It is then easy to justify that y_i (such that $y_i(X_i) = Y_i$) is bounded (Assumption 2.1).

205 **3.1.2. Assumptions on the distribution ν .** We assume a condition on the
 206 probability measure ν and the Markov chain X , which ensures a suitable stability in
 207 the propagation of errors.

208 ASSUMPTION 3.2 (norm-stability). *There exists a constant $\underline{C}_{(3.1)} \geq 1$ such that*
 209 *for any $\varphi : \mathbb{R}^d \mapsto \mathbb{R} \in L^2(\nu)$ and any $0 \leq i \leq N - 1$, we have*

210 (3.1)
$$\int_{\mathbb{R}^d} \mathbb{E} [\varphi^2(X_{i+1}^{i,x})] \nu(dx) \leq \underline{C}_{(3.1)} \int_{\mathbb{R}^d} \varphi^2(x) \nu(dx).$$

211 We now provide some examples of distribution ν where the above assumption
 212 holds, in connection with Examples 2.1, 2.2 and 2.4.

213 PROPOSITION 3.1. Let $\alpha = (\alpha^1, \dots, \alpha^d) \in]0, +\infty[^d$ and assume that $X_{i+1}^{i,x} = x +$
 214 U_i (as in Examples 2.1 and 2.2) with $\mathbb{E} \left[\prod_{j=1}^d e^{\alpha^j |U_i^j|} \right] < +\infty$. Then, the tensor-
 215 product Laplace distribution $\nu(dx) := \prod_{j=1}^d \frac{\alpha^j}{2} e^{-\alpha^j |x^j|} dx$ satisfies Assumption 3.2.

216 *Proof.* The L.H.S. of (3.1) writes

$$217 \quad \mathbb{E} \left[\int_{\mathbb{R}^d} \varphi^2(x + U_i) \nu(dx) \right] = \mathbb{E} \left[\int_{\mathbb{R}^d} \varphi^2(x) \prod_{j=1}^d \frac{\alpha^j}{2} e^{-\alpha^j |x^j - U_i^j|} dx \right]$$

$$218 \quad \leq \mathbb{E} \left[\int_{\mathbb{R}^d} \varphi^2(x) \prod_{j=1}^d \frac{\alpha^j}{2} e^{-\alpha^j |x^j| + \alpha^j |U_i^j|} dx \right]$$

219

220 which leads to the announced inequality (3.1) with $\underline{C}_{(3.1)} := \mathbb{E} \left[\prod_{j=1}^d e^{\alpha^j |U_i^j|} \right]$. \square

221 PROPOSITION 3.2. Let $k > 0$ and assume that $X_{i+1}^{i,x} = Dx + U_i$ for a diago-
 222 nal invertible matrix $D := \text{diag}(D^1, \dots, D^d)$ (a form similar to Example 2.4) with
 223 $\mathbb{E} \left[(1 + |U_i|)^{d(k+1)} \right] < +\infty$. Then, the tensor-product Pareto-type distribution $\nu(dx) :=$
 224 $\prod_{j=1}^d \frac{k}{2} (1 + |x^j|)^{-k-1} dx$ satisfies Assumption 3.2.

225 *Proof.* The L.H.S. of (3.1) equals

$$226 \quad \mathbb{E} \left[\int_{\mathbb{R}^d} \varphi^2(Dx + U_i) \nu(dx) \right]$$

$$227 \quad = \mathbb{E} \left[\int_{\mathbb{R}^d} \varphi^2(x) \det(D^{-1}) \prod_{j=1}^d \frac{k}{2} (1 + |(x^j - U_i^j)/D^j|)^{-k-1} dx \right]$$

$$228 \quad (3.2) \quad \leq \int_{\mathbb{R}^d} \varphi^2(x) \det(D^{-1}) \prod_{j=1}^d \frac{k}{2} \left(\mathbb{E} \left[(1 + |(x^j - U_i^j)/D^j|)^{-d(k+1)} \right] \right)^{1/d} dx.$$

229

On the set $\{|U_i^j| \leq |x^j|/2\}$ we have $(1 + |(x^j - U_i^j)/D^j|) \geq (1 + (|x^j| - |U_i^j|)/D^j) \geq$
 $(1 + |x^j|/(2D^j))$. On the complementary set $\{|U_i^j| > |x^j|/2\}$, the random variable
 inside the j -th expectation in (3.2) is bounded by 1 and furthermore

$$\mathbb{P} \left(|U_i^j| > |x^j|/2 \right) \leq \frac{\mathbb{E} \left[(1 + 2|U_i^j|)^{d(k+1)} \right]}{(1 + |x^j|)^{d(k+1)}}.$$

230 By gathering the two cases, we observe that we have shown that the j -th expectation
 231 in (3.2) is bounded by $\text{Cst}(1 + |x^j|)^{-d(k+1)}$, for any x^j , whence the advertised result. \square

232 *Remarks.*

- 233 • Since we will apply the inequality (3.1) only to functions in a finite dimen-
 234 sional space, the norm equivalence property of finite dimensional space may
 235 also give the existence of a constant $\underline{C}_{(3.1)}$. But the constant built in this
 236 way could depend on the finite dimensional space (and may blow up when its
 237 dimension increases) while here the constant is valid for any φ .
- 238 • The previous examples on ν are related to distributions with independent
 239 components: this is especially convenient when one has to sample ν restricted
 240 to hypercubes \mathcal{H}_k , since we are reduced to independent one-dimensional sim-
 241 ulations.

242 • In Proposition 3.2, had the matrix D been symmetric instead of diagonal, we
243 would have applied an appropriate rotation to the density ν .

244 **3.1.3. Covering number of an approximation space.** To analyze how the
245 M -samples $(X_{i:N}^{i,k,m}, 1 \leq m \leq M)$ from NISR approximates the exact distribution of
246 $X_{i:N}$ with $X_i \stackrel{d}{\sim} \nu_k$ over test functions in the space \mathcal{L}_k , we will invoke concentration of
247 measure inequalities (uniform in \mathcal{L}_k). This is possible thanks to complexity estimates
248 related to \mathcal{L}_k , expressed in terms of covering numbers. Note that the concept of
249 covering numbers is mainly used to introduce Assumption 3.3 and it intervenes in the
250 main theorems only through the proof of Proposition 3.5.

251 We briefly recall the definition of a covering number of a dictionary of functions \mathcal{G} ,
252 see [10, Chapter 9] for more details. For a dictionary \mathcal{G} of functions from \mathbb{R}^d to \mathbb{R} and
253 for M points $x^{1:M} := \{x^{(1)}, \dots, x^{(M)}\}$ in \mathbb{R}^d , an ε -cover ($\varepsilon > 0$) of \mathcal{G} w.r.t. the L^1 -
254 empirical norm $\|g\|_1 := \frac{1}{M} \sum_{m=1}^M |g(x^{(m)})|$ is a finite collection of functions g_1, \dots, g_n
255 such that for any $g \in \mathcal{G}$, we can find a $j \in \{1, \dots, n\}$ such that $\|g - g_j\|_1 \leq \varepsilon$.
256 The smallest possible integer n is called the ε -covering number and is denoted by
257 $\mathcal{N}_1(\varepsilon, \mathcal{G}, x^{1:M})$.

ASSUMPTION 3.3 (Covering the approximation space). *There exist three constants*

$$\alpha_{(3.3)} \geq \frac{1}{4}, \quad \beta_{(3.3)} > 0, \quad \gamma_{(3.3)} \geq 1$$

258 *such that for any $B > 0, \varepsilon \in (0, \frac{4}{15}B]$ and stratum index $1 \leq k \leq K$, the minimal size*
259 *of an ε -covering number of $\mathcal{T}_B \mathcal{L}_k := \{\mathcal{T}_B \varphi : \varphi \in \mathcal{L}_k\}$ is bounded as follows:*

$$260 \quad (3.3) \quad \mathcal{N}_1(\varepsilon, \mathcal{T}_B \mathcal{L}_k, x^{1:M}) \leq \alpha_{(3.3)} \left(\frac{\beta_{(3.3)} B}{\varepsilon} \right)^{\gamma_{(3.3)}}$$

261 *independently of the points sample $x^{1:M}$.*

262 We assume that the above constants do not depend on k , mainly for the sake of
263 simplicity. In the error analysis (see also Proposition A.1), the constants $\alpha_{(3.3)}$ and
264 $\beta_{(3.3)}$ appear in log and thus, they have a small impact on error bounds. On the
265 contrary, $\gamma_{(3.3)}$ appears as a multiplicative factor and we seek to have the smallest
266 estimate.

267 PROPOSITION 3.3. *In the case of approximation spaces \mathcal{L}_k like \mathbf{LP}_0 , \mathbf{LP}_1 or \mathbf{LP}_n ,*
268 *Assumption 3.3 is satisfied with the following parameters: for any given $\eta > 0$, we*
269 *have*

	$\alpha_{(3.3)}$	$\beta_{(3.3)}$	$\gamma_{(3.3)}$
270 \mathbf{LP}_0	1	7/5	1
\mathbf{LP}_1	3	$[4c_\eta 6^\eta]^{1/(1+\eta)} e$	$(d+2)(1+\eta)$
\mathbf{LP}_n	3	$[4c_\eta 6^\eta]^{1/(1+\eta)} e$	$((d+1)^n + 1)(1+\eta)$

271 *where $c_\eta = \sup_{x \geq \frac{45\varepsilon}{2}} x^{-\eta} \log(x)$.*

272 The proof is postponed to the Appendix.

3.2. Main result: Error estimate. We are now in the position to state a
convergence result, expressed in terms of the quadratic error of the best approximation
of y_i on the stratum \mathcal{H}_k :

$$T_{i,k} := \inf_{\varphi \in \mathcal{L}_k} |y_i - \varphi|_{\nu_k}^2 \quad \text{where} \quad |\varphi|_{\nu_k}^2 := \int_{\mathbb{R}^d} |\varphi|^2(x) \nu_k(dx).$$

Our goal is to find an upper bound for the error $\mathbb{E} \left[|y_i^{(M)} - y_i|_\nu^2 \right]$ where

$$|\varphi|_\nu^2 := \int_{\mathbb{R}^d} |\varphi|^2(x) \nu(dx).$$

Note that the above expectation is taken over all the random variables, including the random sources $U^{1:M}$, i.e., we estimate the quadratic error averaged on the root sample.

THEOREM 3.4. *Assume Assumptions 2.2-2.3-3.2-3.3 and define $y_i^{(M)}$ as in Algorithm 1. Then, for any $\varepsilon > 0$, we have*

$$\begin{aligned} \mathbb{E} \left[|y_i^{(M)} - y_i|_\nu^2 \right] &\leq 4(1 + \varepsilon) L_{g_i}^2 \underline{C}_{(3.1)} \mathbb{E} \left[|y_{i+1}^{(M)} - y_{i+1}|_\nu^2 \right] + 2 \sum_{k=1}^K \nu(H_k) T_{i,k} + 4c_{(3.8)}(M) \frac{|y_i|_\infty^2}{M} \\ &\quad + 2\left(1 + \frac{1}{\varepsilon}\right) \frac{\dim(\mathcal{L})}{M} (C_{g_i} + L_{g_i} |y_{i+1}|_\infty)^2 + 8(1 + \varepsilon) L_{g_i}^2 c_{(3.7)}(M) \frac{|y_{i+1}|_\infty^2}{M}. \end{aligned}$$

We emphasize that whenever useful, the constant $4(1 + \varepsilon) L_{g_i}^2 \underline{C}_{(3.1)}$ could be reduced to $(1 + \delta)(1 + \varepsilon) L_{g_i}^2 \underline{C}_{(3.1)}$ (for any given $\delta > 0$) by slightly adapting the proof: namely, the term $4 = 2^2$ comes from two applications of deviation inequalities stated in Proposition A.1. These inequalities are valid with $(1 + \delta)^{\frac{1}{2}}$ instead of 2, up to modifying the constants $c_{(A.2)}(M)$ and $c_{(A.3)}(M)$.

As a very significant difference with usual Regression Monte-Carlo methods (see [9, Theorem 4.11]), in our algorithm there is no competition between the bias term (approximation error) and the variance term (statistical error), while in usual algorithms as the dimension of the approximation space $K \dim(\mathcal{L})$ goes to infinity, the statistical term (of size $\frac{K \dim(\mathcal{L})}{M}$) blows up. This significant improvement comes from the stratification which gives rise to decoupled and low-dimensional regression problems.

Since $y_N^{(M)} = y_N$, we easily derive global error bounds.

COROLLARY 3.1. *Under the assumptions and notations of Theorem 3.4, there exists a constant $C_{(3.4)}(N)$ (depending only on N , $\sup_{0 \leq i < N} L_{g_i}$, $\underline{C}_{(3.1)}$), such that for any $j \in \{0, \dots, N - 1\}$,*

$$\begin{aligned} \mathbb{E} \left[|y_j^{(M)} - y_j|_\nu^2 \right] &\leq C_{(3.4)}(N) \sum_{i=j}^{N-1} \left[\sum_{k=1}^K \nu(H_k) T_{i,k} \right. \\ &\quad \left. + \frac{1}{M} \left(c_{(3.8)}(M) |y_i|_\infty^2 + \dim(\mathcal{L}) (C_{g_i} + L_{g_i} |y_{i+1}|_\infty)^2 + L_{g_i}^2 c_{(3.7)}(M) |y_{i+1}|_\infty^2 \right) \right]. \end{aligned}$$

It is easy to see that if $4(1 + \varepsilon) L_{g_i}^2 \underline{C}_{(3.1)} \leq 1$, then interestingly $C_{(3.4)}(N)$ can be taken uniformly in N . This case corresponds to a small Lipschitz constant of g_i . In the case $4(1 + \varepsilon) L_{g_i}^2 \underline{C}_{(3.1)} \gg 1$, the above error estimates deteriorate quickly as N increases. We shall discuss that in Section 4 which deals with BSDEs and where we propose a different scheme that allows both large Lipschitz constant and large N .

3.3. Proof of Theorem 3.4. Let us start by setting up some useful notations:

$$\begin{aligned} S(x_{i:N}) &:= g_i(y_{i+1}(x_{i+1}), x_{i:N}), & \psi_i^k &:= \text{OLS}(S, \mathcal{L}_k, X_{i:N}^{i,k,1:M}), \\ |f_{i,k,M}^2| &:= \frac{1}{M} \sum_{m=1}^M f^2(X_{i:N}^{i,k,m}) \end{aligned}$$

308

 309 (or $|f|_{i,k,M}^2 := \frac{1}{M} \sum_{m=1}^M f^2(X_i^{i,k,m})$ when f depends only on one argument).

 We first aim at deriving a bound on $\mathbb{E} \left[|y_i^{(M)} - y_i|_{i,k,M}^2 \right]$. First of all, note that

$$|y_i^{(M)} - y_i|_{i,k,M}^2 = \left| \mathcal{T}_{|y_i|_\infty}(\psi_i^{(M),k}) - \mathcal{T}_{|y_i|_\infty}(y_i) \right|_{i,k,M}^2 \leq |\psi_i^{(M),k} - y_i|_{i,k,M}^2$$

310 since the truncation operator is 1-Lipschitz. Now we define

311 (3.5)
$$\mathbb{E} \left[S(X_{i:N}^{i,k,m}) | X_i^{i,k,1:M} \right] = \mathbb{E} \left[S(X_{i:N}^{i,k,m}) | X_i^{i,k,m} \right] = y_i(X_i^{i,k,m})$$

 312 where the first equality is due to the independence of the paths $(X_{i:N}^{i,k,m}, 1 \leq m \leq M)$
 313 (Proposition 2.1) and where the last equality stems from the definition of y_i .

According to [9, Proposition 4.12] which allows to interchange conditional expectation and OLS, we have

$$\mathbb{E} \left[\psi_i^k(\cdot) | X_i^{i,k,1:M} \right] = \text{OLS}(y_i, \mathcal{L}_k, X_{i:N}^{i,k,1:M}).$$

 314 Since the expected values $\left(\mathbb{E} \left[\psi_i^k(X_i^{i,k,m}) | X_i^{i,k,1:M} \right] \right)_{1 \leq m \leq M}$ can be seen as the projec-
 315 tions of $(y_i(X_i^{i,k,m}))_{1 \leq m \leq M}$ on the subspace of \mathbb{R}^M spanned by $\{(\varphi(X_i^{i,k,m}))_{1 \leq m \leq M}, \varphi \in$
 316 $\mathcal{L}_k\}$ and $(\psi_i^{(M),k}(X_i^{i,k,m}))_{1 \leq m \leq M}$ is an element in this subspace, Pythagoras theorem
 317 yields

318
$$|\psi_i^{(M),k} - y_i|_{i,k,M}^2 = \left| \psi_i^{(M),k} - \mathbb{E} \left[\psi_i^k(\cdot) | X_i^{i,k,1:M} \right] \right|_{i,k,M}^2 + \left| \mathbb{E} \left[\psi_i^k(\cdot) | X_i^{i,k,1:M} \right] - y_i \right|_{i,k,M}^2$$
 319
$$= \left| \psi_i^{(M),k} - \mathbb{E} \left[\psi_i^k(\cdot) | X_i^{i,k,1:M} \right] \right|_{i,k,M}^2 + \inf_{\varphi \in \mathcal{L}_k} |\varphi - y_i|_{i,k,M}^2.$$

 321 For any given $\phi \in \mathcal{L}_k$, we have

322
$$\mathbb{E} \left[\inf_{\varphi \in \mathcal{L}_k} |\varphi - y_i|_{i,k,M}^2 \right] \leq \mathbb{E} \left[|\phi - y_i|_{i,k,M}^2 \right] = \mathbb{E} \left[\frac{1}{M} \sum_{m=1}^M |\phi(X_i^{i,k,m}) - y_i(X_i^{i,k,m})|^2 \right]$$
 323
$$= \int_{\mathbb{R}^d} |\phi(x) - y_i(x)|^2 \nu_k(dx).$$
 324

 Taking the infimum over all functions ϕ on the R.H.S. gives

$$\mathbb{E} \left[\inf_{\varphi \in \mathcal{L}_k} |\varphi - y_i|_{i,k,M}^2 \right] \leq T_{i,k}.$$

 325 So, for any $\varepsilon > 0$, we have

326
$$\mathbb{E} \left[|\psi_i^{(M),k} - y_i|_{i,k,M}^2 \right] \leq T_{i,k} + (1 + \varepsilon) \mathbb{E} \left[|\psi_i^{(M),k} - \psi_i^k|_{i,k,M}^2 \right]$$
 327
$$+ (1 + \frac{1}{\varepsilon}) \mathbb{E} \left[\left| \psi_i^k - \mathbb{E} \left[\psi_i^k(\cdot) | X_i^{i,k,1:M} \right] \right|_{i,k,M}^2 \right].$$
 328

 329 By [9, Proposition 4.12], the last term is bounded by $\frac{\dim(\mathcal{L})}{M} (C_{g_i} + L_{g_i} |y_{i+1}|_\infty)^2$ where
 330 $(C_{g_i} + L_{g_i} |y_{i+1}|_\infty)^2$ clearly bounds the conditional variance of $S(X_{i:N}^{i,k})$. This is the

331 statistical error contribution. Here, we have used the independence of $(X_{i:N}^{i,k,m}, 1 \leq$
 332 $m \leq M)$ (Proposition 2.1).

333 The control of the term $\mathbb{E} \left[|\psi_i^{(M),k} - \psi_i^k|_{i,k,M}^2 \right]$ is possible due to the linearity and
 334 stability of OLS [9, Proposition 4.12]:

$$335 \quad |\psi_i^{(M),k} - \psi_i^k|_{i,k,M}^2 \leq |S^{(M)} - S|_{i,k,M}^2 \leq L_{g_i}^2 \frac{1}{M} \sum_{m=1}^M (y_{i+1}^{(M)} - y_{i+1})^2 (X_{i+1}^{i,k,m}),$$

337 where we have taken advantage of the Lipschitz property of g_i w.r.t. the component
 338 y_{i+1} . So far we have shown

$$339 \quad \mathbb{E} \left[|y_i^{(M)} - y_i|_{i,k,M}^2 \right] \leq T_{i,k} + (1 + \varepsilon) L_{g_i}^2 \mathbb{E} \left[\frac{1}{M} \sum_{m=1}^M (y_{i+1}^{(M)} - y_{i+1})^2 (X_{i+1}^{i,k,m}) \right]$$

$$340 \quad (3.6) \quad + (1 + \frac{1}{\varepsilon}) \frac{\dim(\mathcal{L})}{M} (C_{g_i} + L_{g_i} |y_{i+1}|_\infty)^2.$$

342 This shows a relation between the errors at time i and time $i + 1$, but measured in
 343 different norms. In order to retrieve the same $L^2(\nu)$ -norm and continue the analysis,
 344 we will use the norm-stability property (Assumption 3.2) and the following result
 345 about concentration of measures. The proof is a particular case of Proposition A.1 in
 346 the Appendix, with $\psi(x) = (-2|y_{i+1}|_\infty \vee x \wedge 2|y_{i+1}|_\infty)^2$, $B = |y_{i+1}|_\infty$, $\mathcal{K} = \mathcal{L}_k$, $\eta =$
 347 y_{i+1} .

348 PROPOSITION 3.5. Define $(c_{(3.7)}(M), c_{(3.8)}(M))$ by considering $(c_{(A.2)}(M), c_{(A.3)}(M))$
 349 from Proposition A.1 with the values $(\alpha_{(3.3)}, \beta_{(3.3)}, \gamma_{(3.3)})$ instead of (α, β, γ) . Then
 350 we have

$$351 \quad \mathbb{E} \left[\frac{1}{M} \sum_{m=1}^M (y_{i+1}^{(M)} - y_{i+1})^2 (X_{i+1}^{i,k,m}) \right] \leq 2\mathbb{E} \left[|y_{i+1}^{(M)}(X_{i+1}^{i,\nu_k}) - y_{i+1}(X_{i+1}^{i,\nu_k})|^2 \right]$$

$$352 \quad (3.7) \quad + 4c_{(3.7)}(M) \frac{|y_{i+1}|_\infty^2}{M},$$

$$353 \quad (3.8) \quad \mathbb{E} \left[|y_i^{(M)} - y_i|_{\nu_k}^2 \right] \leq 2\mathbb{E} \left[|y_i^{(M)} - y_i|_{i,k,M}^2 \right] + 4c_{(3.8)}(M) \frac{|y_i|_\infty^2}{M}.$$

355 Multiply both sides of Equation (3.7) by $\nu(H_k)$, sum over k , and use the norm-stability
 356 property (Assumption 3.2): it readily follows that

$$357 \quad \sum_{k=1}^K \nu(H_k) \mathbb{E} \left[\frac{1}{M} \sum_{m=1}^M (y_{i+1}^{(M)} - y_{i+1})^2 (X_{i+1}^{i,k,m}) \right]$$

$$358 \quad \leq 2\mathbb{E} \left[|y_{i+1}^{(M)}(X_{i+1}^{i,\nu}) - y_{i+1}(X_{i+1}^{i,\nu})|^2 \right] + 4c_{(3.7)}(M) \frac{|y_{i+1}|_\infty^2}{M}$$

$$359 \quad \leq 2C_{(3.1)} \mathbb{E} \left[|y_{i+1}^{(M)} - y_{i+1}|_\nu^2 \right] + 4c_{(3.7)}(M) \frac{|y_{i+1}|_\infty^2}{M}.$$

361 Similarly, we can get from Equation (3.8) that

$$362 \quad \mathbb{E} \left[|y_i^{(M)} - y_i|_\nu^2 \right] \leq 2 \sum_{k=1}^K \nu(H_k) \mathbb{E} \left[|y_i^{(M)} - y_i|_{i,k,M}^2 \right] + 4c_{(3.8)}(M) \frac{|y_i|_\infty^2}{M}.$$

363

364 Finally, by combining the above estimates with (3.6), we get

$$\begin{aligned}
365 \quad \mathbb{E} \left[|y_i^{(M)} - y_i|_\nu^2 \right] &\leq 4c_{(3.8)}(M) \frac{|y_i|_\infty^2}{M} + 2 \sum_{k=1}^K \nu(H_k) T_{i,k} + 2 \left(1 + \frac{1}{\varepsilon}\right) \frac{\dim(\mathcal{L})}{M} (C_{g_i} + L_{g_i} |y_{i+1}|_\infty)^2 \\
366 \quad &+ 2(1 + \varepsilon) L_{g_i}^2 \left(2C_{(3.1)} \mathbb{E} \left[|y_{i+1}^{(M)} - y_{i+1}|_\nu^2 \right] + 4c_{(3.7)}(M) \frac{|y_{i+1}|_\infty^2}{M} \right). \\
367
\end{aligned}$$

368 This links $\mathbb{E} \left[|y_i^{(M)} - y_i|_\nu^2 \right]$ with $\mathbb{E} \left[|y_{i+1}^{(M)} - y_{i+1}|_\nu^2 \right]$ as announced.

4. Convergence analysis for the solution of BSDEs with the MDP representation. Let us consider the semi-linear final value problem for a parabolic PDE of the form

$$\begin{cases} \partial_t u(t, x) + \frac{1}{2} \Delta u(t, x) + f(t, u(t, x), x) = 0, & t < 1, x \in \mathbb{R}^d, \\ u(1, x) = g(x). \end{cases}$$

This is a simple form of the Hamilton-Jacobi-Bellman equation of stochastic control problems [13]. Under fairly mild assumptions (see [16] for instance), the solution to the above PDE is related to a Backward Stochastic Differential Equation $(\mathcal{Y}, \mathcal{Z})$ driven by a d -dimensional Brownian motion W . Namely,

$$\mathcal{Y}_t = g(W_1) + \int_t^1 f(s, \mathcal{Y}_s, W_s) ds - \int_t^1 \mathcal{Z}_s dW_s$$

and $\mathcal{Y}_t = u(t, W_t)$, $\mathcal{Z}_t = \nabla u(t, W_t)$. Needless to say, the Laplacian Δ and the process W could be replaced by a more general second order operator and its related diffusion process, and that f could depend on the gradient Z as well. We stick to the above setting which is consistent with this work. Taking conditional expectation reduces to

$$\mathcal{Y}_t = \mathbb{E} \left[g(W_1) + \int_t^1 f(s, \mathcal{Y}_s, W_s) ds \mid W_t \right].$$

369 There are several time discretization schemes of \mathcal{Y} (explicit or implicit Euler
370 schemes, high order schemes [6]) but here we follow the Multi-Step Forward Dynamic
371 Programming (MDP for short) Equation of [9], which allows a better error propagation
372 compared to the One-Step Dynamic Programming Equation:

$$\begin{aligned}
373 \quad Y_i &= \mathbb{E} \left[g_N(X_N) + \frac{1}{N} \sum_{j=i+1}^N f_j(Y_j, X_j, \dots, X_N) \mid X_i \right] = y_i(X_i), \quad 0 \leq i < N. \\
374
\end{aligned}$$

Here, we consider a more general path-dependency on f_j , actually this does not affect the error analysis. In comparison with Algorithm 1, we take

$$g_i(y_{i+1:N}, x_{i:N}) = g_N(x_N) + \frac{1}{N} \sum_{j=i+1}^N f_j(y_j, x_{j:N}).$$

375 In [2] similar discrete BSDEs appear but with an external noise. That corresponds to
376 time-discretization of Backward Doubly SDEs, which in turn are related to stochastic
377 semi-linear PDEs.

378 **4.1. Standing assumptions.** We shall now describe the main assumptions that
 379 are needed in the methodology proposed in this paper.

380 **4.1.1. Assumptions on f_i and g_N .**

381 ASSUMPTION 4.1 (Functions f_i and g_N). *Each f_i is Lipschitz w.r.t. y_i , with Lip-*
 382 *schitz constant L_{f_i} and $C_{f_i} = \sup_{x_{i:N}} |f_i(0, x_{i:N})| < +\infty$. Moreover g_N is bounded.*

383 The reader can easily check that y_i is bounded.

384 **4.1.2. Assumptions on the distribution ν .**

385 ASSUMPTION 4.2 (norm-stability). *There exists a constant $\underline{C}_{(4.1)} \geq 1$ such that*
 386 *for any $\varphi \in L^2(\nu)$ and any $0 \leq i < j \leq N$, we have*

$$387 \quad (4.1) \quad \int_{\mathbb{R}^d} \mathbb{E} \left[\varphi^2(X_j^{i,x}) \right] \nu(dx) \leq \underline{C}_{(4.1)} \int_{\mathbb{R}^d} |\varphi(x)|^2 \nu(dx).$$

388 It is straightforward to extend Propositions 3.1 and 3.2 to fulfill the above assumption.

4.2. Main result: error estimate. We express the error in terms of the best
 local approximation error and the averaged one:

$$T_{i,k} := \inf_{\varphi \in \mathcal{L}_k} |y_i - \varphi|_{\nu_k}^2, \quad \nu(T_{i,\cdot}) := \sum_{k=1}^K \nu(H_k) T_{i,k}.$$

389 In this discrete time BSDE context, Theorem 3.4 becomes the following.

THEOREM 4.1. *Assume Assumptions 2.2-2.3-3.3-4.2 and define $y_i^{(M)}$ as in Algo-*
 390 *rithm 1. Set*

$$\bar{\mathcal{E}}(Y, M, i) := \mathbb{E} \left[|y_i^{(M)} - y_i|_{\nu}^2 \right] = \sum_{k=1}^K \nu(\mathcal{H}_k) \mathbb{E} \left[|y_i^{(M)} - y_i|_{\nu_k}^2 \right].$$

390 Define

$$391 \quad \delta_i = 4c_{(3.8)}(M) \frac{|y_i|_{\infty}^2}{M} + 2\nu(T_{i,\cdot}) + 16 \frac{1}{N} \sum_{j=i+1}^{N-1} L_{f_j}^2 c_{(3.7)}(M) \frac{|y_j|_{\infty}^2}{M} +$$

$$392 \quad + 4 \frac{\dim(\mathcal{L})}{M} \left(|y_N|_{\infty} + \frac{1}{N} \sum_{j=i+1}^N (C_{f_j} + L_{f_j} |y_j|_{\infty}) \right)^2.$$

393 Then, letting $L_f := \sup_j L_{f_j}$, we have

$$395 \quad \bar{\mathcal{E}}(Y, M, i) \leq \delta_i + 8\underline{C}_{(4.1)} L_f^2 \exp \left(8\underline{C}_{(4.1)} L_f^2 \right) \frac{1}{N} \sum_{j=i+1}^{N-1} \delta_j.$$

396 The above general error estimates become simpler when the parameters are uniform
 397 in i .

399 COROLLARY 4.1. *Under the assumptions of Theorem 4.1 and assuming that C_{f_i}, L_{f_i}*
 400 *and $|y_i|_{\infty}$ are bounded uniformly in i and N , there exists a constant $C_{(4.2)}$ (independ-*
 401 *ent of N and of approximation spaces \mathcal{L}_k) such that*

$$(4.2)$$

$$402 \quad \bar{\mathcal{E}}(Y, M, i) \leq C_{(4.2)} \left(\frac{c_{(3.8)}(M) + c_{(3.7)}(M) + \dim(\mathcal{L})}{M} + \nu(T_{i,\cdot}) + \frac{1}{N} \sum_{j=i+1}^{N-1} \nu(T_{j,\cdot}) \right).$$

403

404 We observe that this upper bound is expressed in a quite convenient form to let
405 $N \rightarrow +\infty$ and $K \rightarrow +\infty$. As a major difference with the usual Regression Monte
406 Carlo schemes, the impact of the statistical error (through the parameter M) is not
407 affected by the number K of strata.

408 **4.3. Proof of Theorem 4.1.** We follow the arguments of the proof of Theo-
409 rem 3.4 with the following notation:

$$410 \quad S(x_{i:N}) := g_N(x_N) + \frac{1}{N} \sum_{j=i+1}^N f_j(y_j(x_j), x_{j:N}),$$

$$411 \quad S^{(M)}(x_{i:N}) := g_N(x_N) + \frac{1}{N} \sum_{j=i+1}^N f_j(y_j^{(M)}(x_j), x_{j:N}),$$

$$412 \quad \psi_i^k := \text{OLS}(S, \mathcal{L}_k, X_{i:N}^{i,k,1:M}), \quad \psi_i^{(M),k} := \text{OLS}(S^{(M)}, \mathcal{L}_k, X_{i:N}^{i,k,1:M}).$$

414 The beginning of the proof is similar and we obtain (here, there is no need to optimize
415 ε and we take $\varepsilon = 1$)

$$416 \quad \mathbb{E} \left[|y_i^{(M)} - y_i|_{i,k,M}^2 \right] \leq \mathbb{E} \left[|\psi_i^{(M),k} - y_i|_{i,k,M}^2 \right]$$

$$417 \quad \leq T_{i,k} + 2\mathbb{E} \left[|\psi_i^{(M),k} - \psi_i^k|_{i,k,M}^2 \right] + 2\mathbb{E} \left[\left| \psi_i^k - \mathbb{E} \left[\psi_i^k(\cdot) | X_i^{i,k,1:M} \right] \right|_{i,k,M}^2 \right].$$

419 The last term is a statistical error term, which can be controlled as follows:

$$420 \quad \mathbb{E} \left[\left| \psi_i^k - \mathbb{E} \left[\psi_i^k(\cdot) | X_i^{i,k,1:M} \right] \right|_{i,k,M}^2 \right] \leq \frac{\dim(\mathcal{L})}{M} \left(|y_N|_\infty + \frac{1}{N} \sum_{j=i+1}^N (C_{f_j} + L_{f_j} |y_j|_\infty) \right)^2$$

422 where $(\dots)^2$ is a rough bound of the conditional variance of $S(X_{i:N}^{i,k})$.

423 We handle the control of the term $\mathbb{E} \left[|\psi_i^{(M),k} - \psi_i^k|_{i,k,M}^2 \right]$ as in Theorem 3.4 but
424 the results are different because the dynamic programming equation differs:

$$425 \quad \mathbb{E} \left[|\psi_i^{(M),k} - \psi_i^k|_{i,k,M}^2 \right] \leq \mathbb{E} \left[|S^{(M)} - S|_{i,k,M}^2 \right]$$

$$426 \quad \leq \mathbb{E} \left[\frac{1}{M} \sum_{m=1}^M \left(\frac{1}{N} \sum_{j=i+1}^{N-1} L_{f_j} |y_j^{(M)} - y_j| (X_j^{i,k,m}) \right)^2 \right]$$

$$427 \quad \leq \mathbb{E} \left[\frac{1}{M} \sum_{m=1}^M \frac{1}{N} \sum_{j=i+1}^{N-1} L_{f_j}^2 |y_j^{(M)} - y_j|^2 (X_j^{i,k,m}) \right].$$

429 We multiply the above by $\nu(H_k)$, sum over k , apply the *extended* Proposition 3.5
430 valid also for the problem at hand, and the Assumption 4.2. Then, it follows that

$$431 \quad \sum_{k=1}^K \nu(H_k) \mathbb{E} \left[|\psi_i^{(M),k} - \psi_i^k|_{i,k,M}^2 \right] \leq 2 \frac{1}{N} \sum_{j=i+1}^{N-1} L_{f_j}^2 \left(\underline{C}_{(4.1)} \mathbb{E} \left[|y_j^{(M)} - y_j|_\nu^2 \right] + 2c_{(3.7)}(M) \frac{|y_j|_\infty^2}{M} \right).$$

433 On the other hand, from Equation (3.8) we have

$$434 \quad \mathbb{E} \left[|y_i^{(M)} - y_i|_\nu^2 \right] \leq 2 \sum_{k=1}^K \nu(H_k) \mathbb{E} \left[|y_i^{(M)} - y_i|_{i,k,M}^2 \right] + 4c_{(3.8)}(M) \frac{|y_i|_\infty^2}{M}.$$

435

436 Now collect the different estimates: it writes

$$\begin{aligned}
437 \quad \bar{\mathcal{E}}(Y, M, i) &:= \mathbb{E} \left[|y_i^{(M)} - y_i|_\nu^2 \right] \leq 4c_{(3.8)}(M) \frac{|y_i|_\infty^2}{M} + 2 \sum_{k=1}^K \nu(H_k) T_{i,k} \\
438 \quad &+ 8 \frac{1}{N} \sum_{j=i+1}^{N-1} L_{f_j}^2 \left(\underline{C}_{(4.1)} \mathbb{E} \left[|y_j^{(M)} - y_j|_\nu^2 \right] + 2c_{(3.7)}(M) \frac{|y_j|_\infty^2}{M} \right) + \\
439 \quad &+ 4 \frac{\dim(\mathcal{L})}{M} \left(|y_N|_\infty + \frac{1}{N} \sum_{j=i+1}^N (C_{f_j} + L_{f_j} |y_j|_\infty) \right)^2 \\
440 \quad &:= \delta_i + 8\underline{C}_{(4.1)} \frac{1}{N} \sum_{j=i+1}^{N-1} L_{f_j}^2 \bar{\mathcal{E}}(Y, M, j). \\
441
\end{aligned}$$

442 It takes the form of a discrete Gronwall lemma, which easily allows to derive the
443 following upper bound (see [2, Appendix A.3]):

$$\begin{aligned}
444 \quad \bar{\mathcal{E}}(Y, M, i) &\leq \delta_i + 8\underline{C}_{(4.1)} \frac{1}{N} \sum_{j=i+1}^{N-1} \Gamma_{i,j} L_{f_j}^2 \delta_j, \\
445 \quad \text{where } \Gamma_{i,j} &:= \begin{cases} \prod_{i < k < j} (1 + 8\underline{C}_{(4.1)} \frac{1}{N} L_{f_k}^2), & \text{for } i+1 < j, \\ 1, & \text{otherwise.} \end{cases} \\
446
\end{aligned}$$

447 Using now $L_f = \sup_j L_{f_j}$, we get $\Gamma_{i,j} \leq \exp \left(\sum_{i < k < j} 8\underline{C}_{(4.1)} \frac{1}{N} L_{f_k}^2 \right) \leq \exp(8\underline{C}_{(4.1)} L_f^2)$.
448 This completes the proof.

449 **5. Numerical tests.** We shall now illustrate the methodology in two numerical
450 examples coming from practical problems. The first one concerns a reaction-diffusion
451 PDE connected to spatially distributed populations, whereas the second one deals
452 with a stochastic control problem.

453 **5.1. An Application to Reaction-Diffusion Models in Spatially Dis-**
454 **tributed Populations.** In this section we consider a biologically motivated example
455 to illustrate the strength of the stratified resampling regression methodology pre-
456 sented in the previous sections. We selected an application to spatially distributed
457 populations that evolve under reaction diffusion equations. Besides the theoretical
458 challenges behind the models, it has recently attracted attention due to its impact
459 in the spread of infectious diseases [14, 15] and even to the modeling of Wolbachia
460 infected mosquitoes in the fight of disease spreading *Aedes aegypti* [3, 11].

461 The use of reaction diffusion models to describe the population dynamics of a sin-
462 gle species or genetic trait expanding into new territory dominated by another one goes
463 back to the work of R. A. Fisher [7] and A. Kolmogorov et al. [12]. The mathematical
464 model behind it is known as the (celebrated) Fisher-Kolmogorov-Petrovski-Piscounov
465 (FKPP) equation.

466 In a conveniently chosen scale it takes the form, in dimension 1,

$$467 \quad (5.1) \quad \partial_t u + \partial_x^2 u + au(1-u) = 0, \quad u(T, x) = h(x), \quad x \in \mathbb{R}, t \leq T,$$

468 where $u = u(t, x)$ refers to the proportion of members of an invading species in a
469 spatially distributed population on a straight line. The equation is chosen with time

470 running backwards and as a final value problem to allow direct connection with the
 471 standard probabilistic interpretation.

472 It is well known [1] that for any arbitrary positive C , if we define

473 (5.2)
$$h(x) := \left(1 + C \exp\left(\pm \frac{\sqrt{6a}}{6}x\right)\right)^{-2}$$

then

$$u(t, x) = \left(1 + C \exp\left(\frac{5a}{6}(t - T) \pm \frac{\sqrt{6a}}{6}x\right)\right)^{-2}$$

474 is a traveling wave solution to Equation (5.1). The behavior of $h(x)$ as $x \rightarrow \pm\infty$ is
 475 either one or zero according to the sign chosen inside the exponential. Thus describing
 476 full dominance of the invading species or its absence.

477 The probabilistic formulation goes as follows: Introduce the system, as in Sec-
 478 tion 4,

479
$$dP_s = \sqrt{2}dW_s ,$$

 480
$$dY_s = -f(Y_s)ds + Z_s dW_s, \text{ where } f(x) = ax(1 - x), \text{ and } Z_s = \sqrt{2}\partial_x u(s, P_s)$$

 481
$$Y_T = u(T, P_T) = h(P_T) .$$

482 Then, the process $Y_t = \mathbb{E} \left[Y_T + \int_t^T f(Y_s)ds | P_t \right]$ satisfies $Y_t = u(t, P_t)$.

483 To test the algorithms presented herein, we shall start with the following more
 484 general parabolic PDE

485 (5.3)
$$\partial_t W + \sum_{1 \leq i, j \leq d} A_{ij} \partial_{y_i} \partial_{y_j} W + aW(1 - W) = 0 , t \leq T, \text{ and } y \in \mathbb{R}^d .$$

486 Here, the matrix A is chosen as an arbitrary *positive-definite* constant $d \times d$ matrix.
 487 Furthermore, we choose, for convenience, the final condition

488 (5.4)
$$W(T, y) = h(y' \Sigma^{-1} \theta) ,$$

489 where $\Sigma = \Sigma' = \sqrt{A}$ and θ is arbitrary unit vector. We stress that this special choice
 490 of the final condition has the sole purpose of bypassing the need of solving Equa-
 491 tion (5.4) by other numerical methods for comparison with the present methodology.
 492 Indeed, the fact that we are able to exhibit an explicit solution to Equation (5.3) with
 493 final condition (5.4) allows an easy checking of the accuracy of the method. We also
 494 stress that the method developed in this work does not require an explicit knowledge
 495 of the diffusion coefficient matrix A of Equation (5.3) since we shall make use of the
 496 observed paths. Yet the knowledge of the function $W \mapsto aW(1 - W)$ is crucial.

497 It is easy to see that if $u = u(t, x)$ satisfies Equation (5.1) with final condition
 498 given by Equation (5.2) then

499 (5.5)
$$W(t, y) := u(t, y' \Sigma^{-1} \theta)$$

500 satisfies Equation (5.3) with final condition (5.4).

501 An interpretation of the methodology proposed here is the following: If we were
 502 able to observe the trajectories performed by a small number of free individuals ac-
 503 cording to the diffusion process associated to Equation (5.3), even if we did not know
 504 the explicit form of the diffusion (i.e., we did not have a good calibration of the co-
 505 variance matrix) we could use such trajectories to produce a reliable solution to the
 506 final value problem (5.3) and (5.4).

We firstly present some numerical results in dimension 1 (Tables 1-2-3). We have tested both the one-step (Section 3) and multi-step schemes (Section 4). The final time T is fixed to 1 and we use time discretization $t_i = \frac{i}{N}T, 0 \leq i \leq N$ with $N = 10$ or 20. We divide the real line \mathbb{R} into K subintervals $(I_i)_{1 \leq i \leq K}$ by fixing $A = 25$ and dividing $[-A, A]$ into $K - 2$ equal length intervals and then adding $(-\infty, -A)$ and $(A, +\infty)$. We implement our method by using piecewise constant estimation on each interval. Then finally we get a piecewise constant estimation of $u(0, y)$, noted as $\hat{u}(0, y)$. Then we approximate the squared $L^2(\nu)$ error of our estimation by

$$\sum_{1 \leq k \leq K} |u(0, y_k) - \hat{u}(0, y_k)|^2 \nu(I_k)$$

where y_k is chosen as the middle point of the rectangle if I_k is finite and the boundary point if I_k is infinite. We take $\nu(dx) = \frac{1}{2}e^{-|x|}dx$ and we use the restriction of ν on I_k to sample initial points. The squared $L^2(\nu)$ norm of $u(0, \cdot)$ is around 0.25. Finally remark that the error of our method includes three parts: time discretization error, approximation error due to the use of piecewise constant estimation on hypercubes and statistical error due to the randomness of trajectories. In the following tables, M is the number of trajectories that we use (i.e., the root sample).

We observe in Tables 1 and 3 that the approximation error (visible for small K) contributes much more to the global error for the one-step scheme, compared to the multi-step one. This observation is in agreement with those of [5, 9].

When N gets larger with fixed K and M (Tables 1 and 2), we may observe an increase of the global error for the one-step scheme, this is coherent with the estimates of Corollary 3.1.

	$K = 10$	$K = 20$	$K = 50$	$K = 100$	$K = 200$	$K = 400$
$M = 20$	0.0993	0.0253	0.0038	0.0014	0.0014	0.0019
$M = 40$	0.0997	0.0252	0.0034	9.01e-04	5.16e-04	6.17e-04
$M = 80$	0.0993	0.0249	0.0029	6.15e-04	3.92e-04	3.91e-04
$M = 160$	0.0990	0.0248	0.0029	3.15e-04	1.57e-04	1.71e-04
$M = 320$	0.0990	0.0248	0.0028	2.47e-04	1.02e-04	1.19e-04
$M = 640$	0.0990	0.0246	0.0028	2.26e-04	5.46e-05	4.94e-05

TABLE 1
Average squared L^2 errors with 50 macro runs, $N = 10$, one-step scheme.

	$K = 10$	$K = 20$	$K = 50$	$K = 100$	$K = 200$	$K = 400$
$M = 20$	0.1031	0.0299	0.0073	0.0018	0.0011	0.0012
$M = 40$	0.1031	0.0294	0.0066	0.0014	7.86e-04	7.28e-04
$M = 80$	0.1027	0.0293	0.0065	0.0010	3.18e-04	3.86e-04
$M = 160$	0.1027	0.0294	0.0064	8.91e-04	2.46e-04	1.04e-04
$M = 320$	0.1026	0.0293	0.0064	8.39e-04	1.42e-04	7.03e-05
$M = 640$	0.1027	0.0292	0.0063	8.04e-04	8.16e-05	5.60e-05

TABLE 2
Average squared L^2 errors with 50 macro runs, $N = 20$, one-step scheme.

519

Table 4 below describes numerical results in dimension 2. The final time T is fixed to 1 and we use the time discretization $t_i = \frac{i}{N}T, 0 \leq i \leq N$ with $N = 10$. We divide the real line \mathbb{R} into K subintervals $(I_i)_{1 \leq i \leq K}$ by fixing $A = 25$ and dividing

	$K = 10$	$K = 20$	$K = 50$	$K = 100$	$K = 200$	$K = 400$
$M = 20$	0.0484	0.0066	0.0017	0.0015	0.0011	0.0013
$M = 40$	0.0488	0.0058	8.45e-04	5.81e-04	6.35e-04	5.68e-04
$M = 80$	0.0478	0.0053	4.33e-04	2.96e-04	3.45e-04	4.06e-04
$M = 160$	0.0481	0.0051	2.98e-04	2.23e-04	1.71e-04	1.08e-04
$M = 320$	0.0479	0.0051	1.79e-04	6.48e-05	8.38e-05	1.04e-04
$M = 640$	0.0478	0.0050	1.50e-04	6.49e-05	6.66e-05	5.70e-05

TABLE 3

Average squared L^2 errors with 50 macro runs, $N = 10$, multi-step scheme.

$[-A, A]$ into $K - 2$ equal length intervals and then adding $(-\infty, -A)$ and $(A, +\infty)$. We take $\Sigma = [1, \beta; \beta, 1]$ with $\beta = 0.25$ and $\theta = \frac{[1;1]}{\sqrt{2}}$. We implement our method by using piecewise constant estimation on each finite (or infinite) rectangle $I_i \times I_j$. Then finally we get a piecewise constant estimation of $W(0, y)$, noted as $\hat{W}(0, y)$. Then we approximate the squared $L^2(\nu \otimes \nu)$ error of our estimation by

$$\sum_{1 \leq k_1 \leq K, 1 \leq k_2 \leq K} |W(0, y_{k_1}, y_{k_2}) - \hat{W}(0, y_{k_1}, y_{k_2})|^2 \nu \otimes \nu(I_{k_1} \times I_{k_2})$$

520 where (y_{k_1}, y_{k_2}) is chosen as the middle point of the rectangle if $I_{k_1} \times I_{k_2}$ is finite and
 521 the boundary point if one or both of I_{k_1} and I_{k_2} are infinite. We take $\nu(dx) = \frac{1}{2}e^{-|x|}dx$
 522 and we use the restriction of $\nu \otimes \nu$ on $I_i \times I_j$ to sample initial points. The squared
 523 $L^2(\nu \otimes \nu)$ norm of $W(0, \cdot, \cdot)$ is around 0.25.

	$K = 10$	$K = 20$	$K = 50$	$K = 100$	$K = 200$
$M = 20$	0.0592	0.0167	0.0027	0.0018	0.0010
$M = 40$	0.0588	0.0163	0.0022	5.34e-04	5.00e-04
$M = 80$	0.0588	0.0160	0.0019	3.74e-04	2.98e-04
$M = 160$	0.0586	0.0160	0.0018	3.08e-04	9.16e-05
$M = 320$	0.0586	0.0159	0.0017	1.1e-04	9.24e-05

TABLE 4

Average squared L^2 errors with 50 macro runs, $N = 10$, one-step scheme.

524 As for the previous case in dimension 1, we observe that when K is small, it
 525 is useless to increase M . This is because in such case the approximation error is
 526 dominant. But when K is large enough, the performance of our method improves
 527 when M becomes larger, since this time it is the statistical error which becomes
 528 dominant and larger M means smaller statistical error.

529 In the perspective of a given root sample (M fixed), it is recommended to take K
 530 large: indeed, in agreement with Theorems 3.4 and 4.1, we observe from the numerical
 531 results that the global error decreases up to the statistical error term (depending on
 532 M but not K). In this way, for $M = 20$ (resp. $M = 40$) the relative squared L^2 error
 533 is about 0.4% (resp. 0.22%).

534 **5.2. Travel agency problem: when to offer travels, according to cur-**
 535 **rency and weather forecast....** In this section we illustrate the stratified resampler
 536 methodology in the solution of an optimal investment problem. The underlying model
 537 will have two sources of stochasticity, one related to the weather and the other one
 538 to the exchange rate. The corresponding stochastic processes shall be denoted by X_t^1
 539 and X_t^2 .

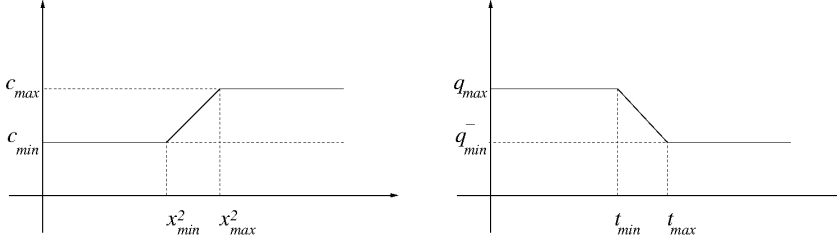


FIG. 2. Pictorial description of the cost function c (left) and of the campaign effectiveness q (right).

540 We envision the following situation: A travel agency wants to launch a campaign
 541 for the promotion of vacations in a warm region abroad during the Fall-Winter season.
 542 Such travel agency would receive a fixed value \underline{c} in local currency from the customers
 543 and on the other hand would have to pay the costs $c = c(\exp(X_{\tau+1/12}^2))$ in a future
 544 time $\tau + 1/12$, where τ is the launching time of the campaign and $X_{\tau+1/12}^2$ is the
 545 prevailing logarithm of exchange rate one month after the launching, with the time
 546 unit set to be one year. The initial time $t = 0$ is by convention October 1st. In other
 547 words, the costs are fixed to the traveler and variable for the agency. A pictorial
 548 description of the cost function is presented in Figure 2.

549 The effectiveness of the campaign will depend on the local temperature $(t -$
 550 $0.25)^2 \times 240 + X_t^1$ (in Celsius) and will be denoted by $q((t - 0.25)^2 \times 240 + X_t^1) \exp(-|t -$
 551 $1/6|)$, where $(t - 0.25)^2 \times 240$ represents the seasonal component and X_t^1 represents
 552 the random part. Its purpose is to capture the idea that if the local temperature
 553 is very low, then people would be more interested in spending some days in a warm
 554 region, whereas if the weather is mild then people would just stay at home. A pictorial
 555 description of the function q is presented in Figure 2. The second part of this function
 556 $\exp(-|t - 1/6|)$ is created to represent the fact that there are likely more registrations
 557 at beginning of December for the period of new year holidays.

558 Thus, our problem consists of finding the function v defined by

$$559 \quad v(X_0^1, X_0^2) = \operatorname{ess\,sup}_{\tau \in \mathcal{T}} \mathbb{E} \left[q((\tau - 0.25)^2 \times 240 + X_\tau^1) e^{-|\tau - 1/6|} \left(\underline{c} - c(e^{X_{\tau+1/12}^2}) \right) \mid X_0^1, X_0^2 \right]$$

$$560 \quad = \operatorname{ess\,sup}_{\tau \in \mathcal{T}} \mathbb{E} \left[q((\tau - 0.25)^2 \times 240 + X_\tau^1) e^{-|\tau - 1/6|} \left(\underline{c} - \mathbb{E} \left[c(e^{X_{\tau+1/12}^2}) \mid X_\tau^2 \right] \right) \mid X_0^1, X_0^2 \right],$$

562 where \mathcal{T} denotes the set of stopping times valued in the weeks of the Fall-Winter
 563 seasons $\{\frac{k}{48}, k = 0, 1, \dots, 24\}$, which corresponds to possible weekly choices for the
 564 travel agency to launch the campaign. The above function v models the optimal
 565 expected benefit for the travel agency and the optimal τ gives the best launching
 566 time. We shall assume, for simplicity, that the processes X^1 and X^2 are uncorrelated
 567 since we do not expect much influence of the weather on the exchange rate or vice-
 568 versa.

569 The problem is tackled by formulating it as a dynamic programming one related
 570 to optimal stopping problems (as exposed in Section 3) using a mean-reversion process
 571 for the underlying process X^1 and a drifted Brownian motion for X^2 . Their dynamics
 572 are given as follows:

$$573 \quad dX_t^1 = -aX_t^1 dt + \sigma_1 dW_t, \quad X_0^1 = 0, \quad X_t^2 = -\frac{\sigma_2^2}{2} t + \sigma_2 B_t.$$

574

	$K = 10$	$K = 20$	$K = 50$	$K = 100$
$M = 20$	0.1827	0.0512	0.0349	0.0269
$M = 40$	0.1982	0.0361	0.0249	0.0114
$M = 80$	0.2063	0.0325	0.0051	0.0047
$M = 160$	0.1928	0.0264	0.0058	0.0067

TABLE 5
Average squared L^2 errors with 20 macro runs. Simple regression.

	$K = 10$	$K = 20$	$K = 50$	$K = 100$
$M = 20$	0.1711	0.0458	0.0436	0.0252
$M = 40$	0.1648	0.0361	0.0130	0.0169
$M = 80$	0.1534	0.0273	0.0109	0.0085
$M = 160$	0.1510	0.0296	0.0048	0.0058

TABLE 6
Average squared L^2 errors with 20 macro runs. Nested regression.

575 The cost function c is chosen piecewise linear so that we can get $\mathbb{E}(c(e^{X_\tau^2+1/12})|X_\tau^2)$
 576 explicitly as a function of X_τ^2 using the Black-Scholes formula in mathematical fi-
 577 nance. Thus we can run our method in two different ways: either using this explicit
 578 expression and apply directly the regression scheme of Section 3; or first estimating
 579 $\mathbb{E}(c(e^{X_\tau^2+1/12})|X_\tau^2)$ by stratified regression then plugging the estimate in our method
 580 again to get a final estimation. We refer to these two different ways as simple regres-
 581 sion and nested regression. The latter case corresponds to a coupled two-component
 582 regression problem (that could be mathematically analyzed very similarly to Sec-
 583 tion 3).

584 The parameter’s values are given as: $a = 2, \sigma_1 = 10, \sigma_2 = 0.2, \underline{c} = 3, x_{min}^2 =$
 585 $e^{-0.5}, x_{max}^2 = e^{0.5}, c_{min} = 1, c_{max} = c_{min} + x_{max}^2 - x_{min}^2, t_{min} = 0, t_{max} = 15, q_{min} =$
 586 $1, q_{max} = 4$. We use the restriction of $\mu(dx) = \frac{k}{2}(1+|x|)^{-k-1}dx$ with $k = 6$ to sample
 587 point for X^1 and the restriction of $\nu(dx) = \frac{1}{2}e^{-|x|}dx$ to sample points for X^2 . Note
 588 that $k = 6$ means that, in the error estimation, more weight is distributed to the
 589 region around $X_0^1 = 0$, which is the real interesting information for the travel agency.

590 We will firstly run our method with $M = 320$ and $K = 300$ to get a reference value
 591 for v then our estimators will be compared to this reference value in a similar way as
 592 in the previous example. The squared $L^2(\mu \otimes \nu)$ norm of our reference estimation is
 593 32.0844. The results are displayed in the Tables 5 and 6.

594 As in Subsection 5.1 and in agreement with Theorem 3.4, we observe an improved
 595 accuracy as K and M increases, independently of each other. The relative error is
 596 rather small even for small M .

597 Interestingly, the nested regression algorithm (which is the most realistic scheme
 598 to use in practice when the model is unknown) is as accurate as the scheme using the
 599 explicit form of the internal conditional expectation $\mathbb{E}[c(e^{X_\tau^2+1/12}) | X_\tau^2]$. Surpris-
 600 ingly, the simple regression scheme takes much more time than the nested regression
 601 one because of the numerous evaluations of the Gaussian CDF in the Black-Scholes
 602 formula.

603 **Appendix A. Appendix.**

A.1. Proof of Proposition 3.3. Consider first the case of the partitioning
 estimate (LP_0) and let $\varepsilon \in (0, \frac{4}{15}B]$. We use an ε -cover in the L^∞ -norm, which

simply reduces to cover $[-B, B]$ with intervals of size 2ϵ . A solution is to take the interval center defined by $h_j = -B + \epsilon + 2\epsilon j$, $0 \leq j \leq n$, where n is the smallest integer such that $h_n \geq B$ (i.e., $n = \lceil \frac{B}{\epsilon} - \frac{1}{2} \rceil$). Thus, we obtain

$$\mathcal{N}_1(\epsilon, \mathcal{T}_B \mathcal{L}_k, x^{1:M}) \leq n + 1 \leq \frac{B}{\epsilon} + \frac{3}{2} \leq \frac{7}{5} \frac{B}{\epsilon}$$

604 where we use the constraint on ϵ .

In the case of general vector space of dimension K , from [10, Lemma 9.2, Theorem 9.4 and Theorem 9.5], we obtain

$$\mathcal{N}_1(\epsilon, \mathcal{T}_B \mathcal{K}, x^{1:M}) \leq 3 \left(\frac{4eB}{\epsilon} \log \left(\frac{6eB}{\epsilon} \right) \right)^{K+1}$$

whenever $\epsilon < B/2$. For ϵ as in the statement of Assumption 3.3, we have $\frac{6eB}{\epsilon} \geq \frac{45e}{2}$. Let $\eta > 0$, since $\log(x) \leq c_\eta x^\eta$ for any $x \geq \frac{45e}{2}$ with $c_\eta = \sup_{x \geq \frac{45e}{2}} \frac{\log(x)}{x^\eta}$, we get

$$\mathcal{N}_1(\epsilon, \mathcal{T}_B \mathcal{K}, x^{1:M}) \leq 3 \left([4c_\eta 6^\eta]^{1/(1+\eta)} \frac{eB}{\epsilon} \right)^{(K+1)(1+\eta)}.$$

605 For \mathbf{LP}_1 and \mathbf{LP}_n , we have respectively $K = d + 1$ and $K = (d + 1)^n$, therefore the
606 announced result. Whenever useful, the choice $\eta = 1$ gives $\beta_{(3.3)} \leq 3.5$.

607 For the partitioning estimate (case \mathbf{LP}_0), we could also use this estimate with
608 $K = 1$ but with the first arguments, we get better parameters (especially for γ).

609 A.2. Probability of uniform deviation.

610 LEMMA A.1 ([9, Lemma B.2]). *Let \mathcal{G} be a countable set of functions $g : \mathbb{R}^d \mapsto$
611 $[0, B]$ with $B > 0$. Let $\mathcal{X}, \mathcal{X}^{(1)}, \dots, \mathcal{X}^{(M)}$ ($M \geq 1$) be i.i.d. \mathbb{R}^d valued random
612 variables. For any $\alpha > 0$ and $\epsilon \in (0, 1)$ one has*

$$\begin{aligned} 613 \quad & \mathbb{P} \left(\sup_{g \in \mathcal{G}} \frac{\frac{1}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)}) - \mathbb{E}[g(\mathcal{X})]}{\alpha + \frac{1}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)}) + \mathbb{E}[g(\mathcal{X})]} > \epsilon \right) \\ 614 \quad & \leq 4\mathbb{E} \left[\mathcal{N}_1 \left(\frac{\alpha\epsilon}{5}, \mathcal{G}, \mathcal{X}^{1:M} \right) \right] \exp \left(- \frac{3\epsilon^2 \alpha M}{40B} \right), \end{aligned}$$

$$\begin{aligned} 615 \quad & \mathbb{P} \left(\sup_{g \in \mathcal{G}} \frac{\mathbb{E}[g(\mathcal{X})] - \frac{1}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)})}{\alpha + \frac{1}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)}) + \mathbb{E}[g(\mathcal{X})]} > \epsilon \right) \\ 616 \quad & \leq 4\mathbb{E} \left[\mathcal{N}_1 \left(\frac{\alpha\epsilon}{8}, \mathcal{G}, \mathcal{X}^{1:M} \right) \right] \exp \left(- \frac{6\epsilon^2 \alpha M}{169B} \right). \end{aligned}$$

618 A.3. Expected uniform deviation.

619 PROPOSITION A.1. *For finite $B > 0$, let $\mathcal{G} := \{\psi(\mathcal{T}_B \phi(\cdot) - \eta(\cdot)) : \phi \in \mathcal{K}\}$, where
620 $\psi : \mathbb{R} \rightarrow [0, \infty)$ is Lipschitz continuous with $\psi(0) = 0$ and Lipschitz constant L_ψ ,
621 $\eta : \mathbb{R}^d \rightarrow [-B, B]$, and \mathcal{K} is a finite K -dimensional vector space of functions with*

$$622 \quad (\text{A.1}) \quad \mathcal{N}_1(\epsilon, \mathcal{T}_B \mathcal{K}, \mathcal{X}^{1:M}) \leq \alpha \left(\frac{\beta B}{\epsilon} \right)^\gamma \quad \text{for } \epsilon \in \left(0, \frac{4}{15} B \right]$$

623 *for some positive constants α, β, γ with $\alpha \geq 1/4$ and $\gamma \geq 1$. Then, for $\mathcal{X}^{(1)}, \dots, \mathcal{X}^{(M)}$
624 i.i.d. copies of \mathcal{X} , we have*

$$625 \quad \mathbb{E} \left[\sup_{g \in \mathcal{G}} \left(\frac{1}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)}) - 2 \int_{\mathbb{R}^d} g(x) \mathbb{P} \circ \mathcal{X}^{-1}(dx) \right)_+ \right] \leq c_{(\text{A.2})}(M) \frac{BL_\Psi}{M}$$

(A.2)

$$626 \quad \text{with } c_{(\text{A.2})}(M) := 120 \left(1 + \log(4\alpha) + \gamma \log \left(\left(1 + \frac{\beta}{16} \right) M \right) \right),$$

$$627 \quad \mathbb{E} \left[\sup_{g \in \mathcal{G}} \left(\int_{\mathbb{R}^d} g(x) \mathbb{P} \circ \mathcal{X}^{-1}(\mathrm{d}x) - \frac{2}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)}) \right)_+ \right] \leq c_{(\text{A.3})}(M) \frac{BL_\Psi}{M}$$

(A.3)

$$628 \quad \text{with } c_{(\text{A.3})}(M) := \frac{507}{2} \left(1 + \log(4\alpha) + \gamma \log \left(\left(1 + \frac{8\beta}{169} \right) M \right) \right).$$

630 *Proof.* The idea is to adapt the arguments of [9, Proposition 4.9].

631 \triangleright We first show (A.2). Set $\mathcal{Z} := \sup_{g \in \mathcal{G}} \left(\frac{1}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)}) - 2 \int_{\mathbb{R}^d} g(x) \mathbb{P} \circ \mathcal{X}^{-1}(\mathrm{d}x) \right)_+$. Let us find an upper bound for $\mathbb{P}(\mathcal{Z} > \varepsilon)$ in order to bound $\mathbb{E}[\mathcal{Z}] = \int_0^\infty \mathbb{P}(\mathcal{Z} > \varepsilon) \mathrm{d}\varepsilon$. Using the equality

$$634 \quad \mathbb{P}(\mathcal{Z} > \varepsilon) = \mathbb{P} \left(\exists g \in \mathcal{G} : \frac{\frac{1}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)}) - \int_{\mathbb{R}^d} g(x) \mathbb{P} \circ \mathcal{X}^{-1}(\mathrm{d}x)}{2\varepsilon + \int_{\mathbb{R}^d} g(x) \mathbb{P} \circ \mathcal{X}^{-1}(\mathrm{d}x) + \frac{1}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)})} > \frac{1}{3} \right),$$

635 and that the elements of \mathcal{G} take values in $[0, 2BL_\psi]$, it follows from Lemma A.1 that

$$636 \quad \mathbb{P}(\mathcal{Z} > \varepsilon) \leq 4\mathbb{E} \left[\mathcal{N}_1 \left(\frac{2\varepsilon}{15}, \mathcal{G}, \mathcal{X}^{1:M} \right) \right] \exp \left(- \frac{\varepsilon M}{120BL_\psi} \right).$$

637 Define $\mathcal{T}_B\mathcal{K}$ as in Proposition A.1. Since $|\psi(\phi_1(x) - \eta(x)) - \psi(\phi_2(x) - \eta(x))| \leq$
638 $L_\psi |\phi_1(x) - \phi_2(x)|$ for all $x \in \mathbb{R}^d$ and all (ϕ_1, ϕ_2) , it follows that

$$639 \quad \mathcal{N}_1 \left(\frac{2\varepsilon}{15}, \mathcal{G}, \mathcal{X}^{1:M} \right) \leq \mathcal{N}_1 \left(\frac{2\varepsilon}{15L_\psi}, \mathcal{T}_B\mathcal{K}, \mathcal{X}^{1:M} \right).$$

640 Due to Equation (A.1), we deduce that

$$641 \quad (\text{A.4}) \quad \mathbb{P}(\mathcal{Z} > \varepsilon) \leq 4\alpha \left(\frac{15\beta BL_\psi}{2\varepsilon} \right)^\gamma \exp \left(- \frac{\varepsilon M}{120BL_\psi} \right)$$

642 whenever $\frac{2\varepsilon}{15L_\psi} \leq \frac{4}{15}B$, i.e., $\varepsilon \leq 2BL_\psi$. On the other hand, $\mathbb{P}(\mathcal{Z} > \varepsilon) = 0$ for all
643 $\varepsilon > 2BL_\psi$. Setting $a = \frac{15\beta BL_\psi}{2}$, $b = \frac{1}{120BL_\psi}$, it follows from (A.4) that

$$644 \quad \mathbb{P}(\mathcal{Z} > \varepsilon) \leq 4\alpha \left(\frac{a}{\varepsilon} \right)^\gamma \exp(-bM\varepsilon), \quad \forall \varepsilon > 0.$$

645 Fix ε_0 to be some finite value (to be determined later) such that

$$646 \quad (\text{A.5}) \quad \varepsilon_0 \geq \frac{a}{M(1+ab)}.$$

647 It readily follows that

$$648 \quad \mathbb{E}[\mathcal{Z}] = \int_0^\infty \mathbb{P}(\mathcal{Z} > \varepsilon) \mathrm{d}\varepsilon \leq \varepsilon_0 + \int_{\varepsilon_0}^\infty 4\alpha \left(\frac{a}{\varepsilon} \right)^\gamma \exp(-bM\varepsilon) \mathrm{d}\varepsilon$$

$$649 \quad \leq \varepsilon_0 + \frac{4\alpha}{bM} (M(1+ab))^\gamma \exp(-bM\varepsilon_0).$$

650

651 We choose $\varepsilon_0 = \frac{1}{bM} \log(4\alpha((1+ab)M)^\gamma)$: It satisfies (A.5) since

$$652 \quad \frac{1}{bM} \log(4\alpha((1+ab)M)^\gamma) \geq \frac{a}{M} \frac{\log(1+ab)}{ab} \geq \frac{a}{M} \frac{1}{1+ab}$$

654 (use $\alpha \geq 1/4$, $\gamma \geq 1$, $M \geq 1$ and $\log(1+x) \geq x/(1+x)$ for all $x \geq 0$). Moreover, this
655 choice of ε_0 implies that

$$656 \quad (\text{A.6}) \quad \mathbb{E}[\mathcal{Z}] \leq \frac{1}{bM} \left(1 + \log(4\alpha) + \gamma \log((1+ab)M) \right) \\ 657 \quad = \frac{120BL_\psi}{M} \left(1 + \log(4\alpha) + \gamma \log\left(1 + \frac{\beta}{16}M\right) \right). \\ 658$$

659 The inequality (A.2) is proved.

660 \triangleright We now justify (A.3) by similar arguments. Set $\mathcal{Z} := \sup_{g \in \mathcal{G}} \left(\int_{\mathbb{R}^d} g(x) \mathbb{P} \circ \right.$
661 $\left. \mathcal{X}^{-1}(dx) - \frac{2}{M} \sum_{m=1}^M g(\mathcal{X}^{(m)}) \right)_+$. From Lemma A.1, we get

$$662 \quad \mathbb{P}(\mathcal{Z} > \varepsilon) \leq 4\mathbb{E} \left[\mathcal{N}_1\left(\frac{\varepsilon}{12}, \mathcal{G}, \mathcal{X}^{1:M}\right) \right] \exp\left(-\frac{2\varepsilon M}{507BL_\psi}\right).$$

663 Since $\mathcal{N}_1\left(\frac{\varepsilon}{12}, \mathcal{G}, \mathcal{X}^{1:M}\right) \leq \mathcal{N}_1\left(\frac{\varepsilon}{12L_\psi}, \mathcal{T}_B\mathcal{K}, \mathcal{X}^{1:M}\right)$ and thanks to (A.1), we derive

$$664 \quad (\text{A.7}) \quad \mathbb{P}(\mathcal{Z} > \varepsilon) \leq 4\alpha \left(\frac{12\beta BL_\psi}{\varepsilon} \right)^\gamma \exp\left(-\frac{2\varepsilon M}{507BL_\psi}\right)$$

665 whenever $\frac{\varepsilon}{12L_\psi} \leq \frac{4}{15}B$. For other values of ε the above probability is zero, therefore
666 (A.7) holds for any $\varepsilon > 0$. The end of the computations is now very similar to the
667 previous case: we finally get the inequality (A.6) for the new \mathcal{Z} with adjusted values
668 $a = 12\beta BL_\psi$, $b = \frac{2}{507BL_\psi}$. Thus inequality (A.3) is thus proved. \square

669

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