

Stress gradient effects on the nucleation and propagation of cohesive cracks

Tuan Hiep Pham, Jérôme Laverne, Jean-Jacques Marigo

▶ **To cite this version:**

Tuan Hiep Pham, Jérôme Laverne, Jean-Jacques Marigo. Stress gradient effects on the nucleation and propagation of cohesive cracks. Discrete and Continuous Dynamical Systems - Series S, 2016, 9 (2), pp.557 - 584. 10.3934/dcdss.2016012 . hal-01345476

HAL Id: hal-01345476

https://polytechnique.hal.science/hal-01345476

Submitted on 18 Jul 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

STRESS GRADIENT EFFECTS ON THE NUCLEATION AND THE PROPAGATION OF COHESIVE CRACKS

TUAN-HIEP PHAM, JÉRÔME LAVERNE, AND JEAN JACQUES MARIGO

Abstract. The aim of the present work is to study the nucleation and propagation of cohesive cracks in two-dimensional elastic structures. The evolution of the crack is governed by Dugdale's cohesive force model. Specifically, we investigate the stabilizing effect of the non-uniformity of the stress field by introducing a length ℓ which characterizes the stress gradient in the neighborhood of the point where the crack nucleates. We distinguish two stages in the crack evolution: the first one where all the crack is submitted to cohesive forces, followed by a second one where a non cohesive part appears. Assuming that the material characteristic length d_c associated with Dugdale's model is small by comparison to the dimension L of the body, we develop a two-scale approach, and using the complex analysis method, we obtain the entire evolution of the crack with the loading in a closed form. In particular, we show that the propagation is stable during the first stage, but becomes unstable with a brutal jump of the crack length as soon as the non cohesive crack part appears. We discuss also the influence of all the parameters of the problem and study the sensitivity to imperfections.

1. Introduction

1.1. **Cohesive force models vs Griffith's model: the state of the art.** Griffith's theory of fracture [Griffith, 1920] is based on the concept of critical energy release rate G_c which comes from the fundamental but somewhat too restrictive assumption that the surface energy associated with a crack is proportional to the area of the crack (at least in a homogeneous and isotropic body), or, equivalently, that there is no interaction between the lips of a crack. It remains the most used approach in fracture mechanics thanks to its simplicity in terms of material behavior. However, this theory contains some major drawbacks. In particular, since Griffith's model does not contain a critical stress, (i) it allows stress singularity and (ii) cannot give an account of the nucleation of crack in a sound body. Accordingly, cohesive-force models have been introduced with main goal to prohibit these types of unphysical singularities by allowing finite stresses only. Specifically, following the ideas of [Dugdale, 1960] and [Barenblatt, 1962], many such models have been proposed and tested, see for instance[Tvergaard, 1990; Needleman, 1992; Keller et al., 1999; Roe and Siegmund, 2002; Talon and Curnier, 2003; Del Piero and Raous, 2010].

In the first stage of their developments, these cohesive models have been used in the restricted framework where the body contains preexisting cracks without considering the issue of their nucleation. In this context, various comparisons between Griffith and Barenblatt models were carried out to establish precise contributions of the latter one with respect to the former. In particular, under monotonic loading which does not require to introduce any irreversibility condition, it was rigorously proved, first by [Marigo and Truskinovsky, 2004] in a restricted 1D setting, then by [Giacomini, 2005] in 3D, that Barenblatt's model leads to a law of crack propagation which converges (in the sense of Gamma-convergence) to the Griffith law when, at given G_c, the ratio between the material characteristic length (which is necessarily present in Barenblatt's model) and the size of the body goes to 0, see also [Willis, 1967] for a more formal proof. In essence this result means that the role of cohesive forces becomes negligible, as far as the crack propagation only is concerned, once the length of the crack is sufficiently large. In such a case, the cohesive zone is essentially concentrated in the neighborhood of the crack tip, its size being of the order of the material characteristic length. In fact,

Date: version of May 12, 2015.

the cohesive forces are then useful just to correct the shape of the crack opening near the crack tip by killing any stress singularity.

A first fundamental difference between the two models appears when one uses them in the case of cycling loading. Indeed, with Griffith's model a crack can propagate during the first cycle only and no more after. On the other hand, after introducing a suitable irreversibility condition in Barenblatt's type model, like in the models proposed by [Roe and Siegmund, 2002; Talon and Curnier, 2003; Jaubert and Marigo, 2006], a crack can evolve from one cycle to the other and hence it becomes possible to give an account of the fatigue phenomenon. Moreover, one can prove that the fatigue law induced by the cohesive force model converges to a Paris-like fatigue law [Paris et al., 1961] when the material characteristic length is small by comparison to the size of the body. This fundamental result was first observed in [Nguyen et al., 2001] by purely numerical considerations before to be rigorously proved with Gamma-convergence argument by [Jaubert and Marigo, 2006; Bourdin et al., 2008] in the restricted context of a peeling test. Its generalization to cracks in mode I or III is briefly presented in [Abdelmoula et al., 2009b; Abdelmoula et al., 2009a] and detailed in [Abdelmoula et al., 2010].

1.2. **The issue of the nucleation of cracks with cohesive force models.** Furthermore, Barenblatt's model radically differs from Griffith's model as far as the nucleation of cracks is concerned. Indeed, it turns out that the cohesive models, because they contain a critical stress, are also able to explain the process of nucleation of a crack in a sound body whereas Griffith's model cannot, in general. The first author who was able to establish such a result in a rigorous manner was [Del Piero, 1999] in a one-dimensional setting by using a variational approach and introducing a stability criterion. With such a criterion, a complete comparison between the two types of models can be carried out in the 1D restricted setting, see [Charlotte et al., 2000]. Specifically, assuming that the surface energy density which governs the cohesive forces is a smooth increasing concave function $\Phi([\![u]\!])$ of the displacement jump $[\![u]\!]$, then the stability criterion requires that the stress field $\sigma(x)$ at equilibrium be everywhere less than the derivative at 0 of the energy density function, say $\sigma(x) \leq \sigma_c := \Phi'(0)$. Therefore, σ_c plays the role of the material critical stress. This result can be extended to a general 3D setting, still by using the stability criterion. In particular, assuming that the material is isotropic and hence that the surface energy density is only a function of the normal jump of the displacement and of the norm of the tangential jump of the displacement across the lips of the crack, say $\Phi([\![\mathbf{u}]\!] \cdot \mathbf{n}, \|[\![\mathbf{u}]\!] - ([\![\mathbf{u}]\!] \cdot \mathbf{n})\ \mathbf{n}\|)$ where $\mathbf{n}$ denotes the local unit normal vector to the crack, it is stated in [Laverne and Marigo, 2004] and proved in [Charlotte et al., 2006] that the criterion of nucleation of a crack takes the form of an intrinsic curve in the Mohr stress plane which involves the directional derivatives at $(0,0)$ of Φ. Furthermore, when Φ admits partial derivatives at $(0,0)$, the nucleation criterion simply reduces to the two usual criteria of maximal shear stress and maximal tensile stress. Specifically, that means that everywhere in the body the stress field must satisfy the two following inequalities:

$$\max_{\mathbf{n}:\|\mathbf{n}\|=1} \left\|\boldsymbol{\sigma}\mathbf{n} - \sigma_{nn}\mathbf{n}\right\| \leq \tau_c, \qquad \max_{\mathbf{n}:\|\mathbf{n}\|=1} \sigma_{nn} \leq \sigma_c$$

where τ_c and σ_c denotes respectively the maximal shear stress and the maximal normal stress that the material can sustain. This result is really fundamental by establishing a link between the nucleation of cohesive cracks and the empirical criteria of strength of materials proposed by the engineers, like Mohr and Caquot, at the early of the twentieth century.

However, this result only says that a cohesive crack will appear somewhere in the body when the stress field predicted by a pure elastic response reaches a threshold, but it says nothing on the growth process of these nucleated cracks. To treat this delicate issue, one must include in a unique formulation both the nucleation and the propagation of (cohesive) cracks. In essence, that is one of the main purposes of the variational approach to fracture, see [Bourdin et al., 2008] for an overview. In this context of cohesive force models, some partial results have already been obtained. For instance, [Ferdjani et al., 2007; Ferdjani et al., 2009] study the size and shape effects of preexisting defects in the case of Dugdale's model. It is in particular shown that the value of the loading at which the first cohesive crack occurs strongly depends on the shape

of the preexisting defect. On the other hand, sufficiently small defects have practically no influence on the overall resistance of a structure.

Besides these first results on shape or size effects, [Dang et al., 2013; Dang et al., 2014] treat the problem of the nucleation and the propagation of a cohesive crack at the tip of a notch, still in the context of Dugdale's model. In such a situation, since the notch would induce a singularity of the stress if the response was purely elastic, there exists no elastic phase in the loading process, a cohesive crack is created at the tip of the notch as soon as a load is applied. The length of the crack and its opening grow in such a manner that there is no singularity of the stresses. During a first stage of the loading, the growth of the crack is stable, but it becomes unstable when the opening of the crack just at the tip of the notch reaches the critical value δ_c associated with Dugdale's model. Then a macroscopic crack is created by instability, the length of which being governed by a condition of energy conservation. In the case where the material characteristic length is small by comparison to the overall dimension of the body, it is even possible by using a two-scale technique to obtain in a closed form the formula giving the load at which a macroscopic appears at the notch tip.

1.3. **The nucleation and growth of a crack at a regular point.** The goal of the present work is to consider the same problem as in [Dang et al., 2013; Dang et al., 2014] but for another type of structure. Specifically, we assume that the body contains neither a notch nor any corner which would induce elastic singularities. In other words the stress field associated with a pure elastic response is assumed to be smooth and bounded, but non uniform. Considering a symmetric structure submitted to an increasing loading and adopting Dugdale's law as the cohesive force model, a cohesive crack appears at a material point where the normal stress is maximal when the loading reaches a critical value. The question is then to study the process of propagation of this crack and to highlight the stabilizing effects of the stress gradients. Specifically, one shows that the growth of the crack is first progressive, which means that it depends continuously on the loading parameter, by virtue of the non-uniformity of the stress field. In fact, the first stage of the crack growth is controlled by the second derivatives of the stress field. In a second times, when the loading reaches a value such that the opening of the crack at its center reaches the critical value δ_c, then a non cohesive zone appears in the center and the propagation becomes brutal, the size of the crack jumping instantaneously to a value which is fixed by the characteristic length of the stress gradient. This second critical value of the loading can be seen as the moment where a genuinely macroscopic and non cohesive crack appears. The main feature of the paper is to obtain all the results in a closed form by using the method of complex potentials and a two-scale technique.

Specifically the paper is organized as follows. The section 2 is devoted to the setting of the problem and to the main assumptions whereas the section 3 contains its resolution and the major part of the results. In particular, we construct a solution in a closed form by using a two-scale approach and the method of complex potentials. This latter section finishes by a long discussion where one studies the influence of the parameters, specially the dependence on the material length d_c and the stress gradient length ℓ, and the sensibility of the response to the imperfections. A short comparison with Griffith's theory is also presented and, finally, the main ingredients of the resolution by using complex potentials are recalled in the appendix.

2. Setting of the problem and main assumptions

2.1. **The body, its elastic behavior and its loading.** Throughout the paper the analysis is made in a plane strain setting. One uses a Cartesian system (x_1, x_2, x_3) with its canonical orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. The reference configuration of the body is the open subset Ω of $\mathbb{R}^2$ in the plane (x_1, x_2). The body is made of an isotropic brittle material whose elastic behavior before cracking is characterized by its Lamé coefficients λ and μ (or equivalently by its Young modulus E and its Poisson ratio ν). This material can be damaged by cracks the behavior of which is governed by the Dugdale model (see below for a precise statement of that model). The body is submitted to a proportional loading parameterized by the increasing parameter $t > 0$ called from now the time. Accordingly, if the response were purely elastic, then the displacement field $\mathbf{u}^{\rm el}(t)$

and the stress fields $\boldsymbol{\sigma}^{\mathrm{el}}(t)$ at time t would be the solutions of the following linear boundary value problem

$$\begin{cases} \mathrm{div}\boldsymbol{\sigma}^{\mathrm{el}}(t) = 0 \quad \text{in} \quad \Omega \\ \boldsymbol{\sigma}^{\mathrm{el}}(t) = \lambda \mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{el}}(t)))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{el}}(t)) \quad \text{in} \quad \Omega \\ \boldsymbol{\sigma}^{\mathrm{el}}(t)\mathbf{n} = t\mathbf{F} \quad \text{on} \quad \partial_N\Omega \\ \mathbf{u}^{\mathrm{el}}(t) = t\mathbf{U} \quad \text{on} \quad \partial_D\Omega \end{cases} \tag{1}$$

In (1), $\boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{el}}(t))$ denotes the strain field associated with the displacement field $\mathbf{u}^{\mathrm{el}}(t)$, *i.e.* the symmetric part of the gradient of $\mathbf{u}^{\mathrm{el}}(t)$. The body forces are neglected, $\partial_N\Omega$ represents the part of the boundary where the surface forces are (progressively) applied whereas $\partial_D\Omega$ represents the complementary part of the boundary where the displacements are prescribed. The loading is proportional in the sense that the intensity of the applied forces and the amplitude of the prescribed displacements are proportional to the loading parameter t.

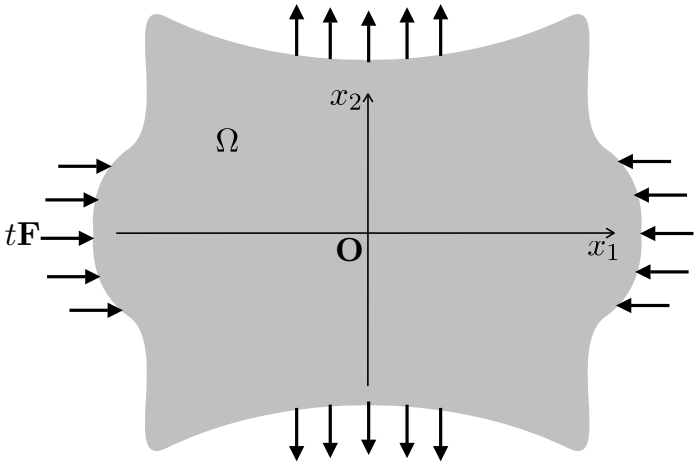


FIGURE 1. The body and its loading.

By virtue of the linearity of the problem (1), its solution depends linearly on t and hence can be written

$$\mathbf{u}^{\mathrm{el}}(t) = t\mathbf{u}^{\mathrm{el}}, \qquad \boldsymbol{\sigma}^{\mathrm{el}}(t) = t\boldsymbol{\sigma}^{\mathrm{el}}, \tag{2}$$

where $(\mathbf{u}^{\mathrm{el}}, \boldsymbol{\sigma}^{\mathrm{el}})$ are solutions of the following linear elastic problem

$$\begin{cases} \mathrm{div}\boldsymbol{\sigma}^{\mathrm{el}} = 0 \quad \text{in} \quad \Omega \\ \boldsymbol{\sigma}^{\mathrm{el}} = \lambda \mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{el}}))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{el}}) \quad \text{in} \quad \Omega \\ \boldsymbol{\sigma}^{\mathrm{el}}\mathbf{n} = \mathbf{F} \quad \text{on} \quad \partial_N\Omega \\ \mathbf{u}^{\mathrm{el}} = \mathbf{U} \quad \text{on} \quad \partial_D\Omega \end{cases} \tag{3}$$

2.2. **Symmetry and smoothness assumptions.** We assume that the body is symmetric with respect to the two axes $x_1 = 0$ and $x_2 = 0$. Moreover, the loading preserves this symmetry and the elastic response enjoys the following properties:

(1) The shear stress $\sigma^{\mathrm{el}}_{12}$ vanishes on the axes $x_1 = 0$ and $x_2 = 0$. Consequently, the stress tensor is diagonal in the basis $(\mathbf{e}_1, \mathbf{e}_2)$ at each point of the axes and its eigenvalues are respectively denoted σ^{el}_1 and σ^{el}_2;
(2) The elastic stress field $\boldsymbol{\sigma}^{\mathrm{el}}(\mathbf{x})$ is a smooth function of $\mathbf{x}$. The maximum of $\sigma^{\mathrm{el}}_{nn}(\mathbf{x})$ is reached at the origin $\mathbf{O} = (0, 0)$, in the direction $\mathbf{n} = \mathbf{e}_2$ and is positive.

These symmetry and smoothness assumptions on the elastic stress field induce some properties on the repartition of the stresses along the axis $x_2 = 0$ that will be useful in the sequel. Specifically,

(1) the stress vector $\boldsymbol{\sigma}^{\rm el}(x_1, 0)\mathbf{e}_2$ is purely normal, say

$$\boldsymbol{\sigma}^{\rm el}(x_1, 0)\mathbf{e}_2 = \Sigma(x_1)\mathbf{e}_2. \tag{4}$$

(2) the normal stress distribution $\Sigma(x_1)$ is an even smooth function of x_1 which is maximum at $x_1 = 0$. Therefore, if we expand it with respect to x_1, near $x_1 = 0$, up to the second order, we obtain

$$\Sigma(x_1) = \sigma_2^{\rm el}(0, 0) + \frac{1}{2}\sigma_{2,11}^{\rm el}(0, 0)x_1^2 + o(x_1^2)$$

where the normal stress at the origin $\sigma_2^{\rm el}(0, 0)$ is positive and its second derivative $\sigma_{2,11}^{\rm el}(0, 0)$ is negative.

(3) Introducing the characteristic length ℓ of the stress gradient,

$$\ell := 2\sqrt{\frac{\sigma_2^{\rm el}(0, 0)}{-\sigma_{2,11}^{\rm el}(0, 0)}}, \tag{5}$$

the expansion of the normal stress distribution can read as

$$\Sigma(x_1) = \sigma_2^{\rm el}(0, 0)\left(1 - \frac{2x_1^2}{\ell^2}\right) + o(x_1^2). \tag{6}$$

This expansion up to the second order can be considered as a good approximation of $\Sigma(x_1)$ provided that x_1 is small by comparison to ℓ.

Remark 1. *If one considers that the expansion of the normal stress distribution up to the second order is the true stress distribution whatever x_1, i.e. if $\Sigma(x_1)$ is given by*

$$\Sigma(x_1) = \sigma_2^{\rm el}(0, 0)\left(1 - \frac{2x_1^2}{\ell^2}\right),$$

then the normal stress is maximal at 0, positive (in tension) for $|x_1| < \ell/\sqrt{2}$ and negative (in compression) for $|x_1| > \ell/\sqrt{2}$. That presence of a compression at a large distance of the origin will limit the propagation of a crack as we will see in Section 3.2.

2.3. **Dugdale's model of crack opening.** The nucleation and the growth of cracks in the body are governed by Dugdale's cohesive force model whose main ingredients are recalled here. This model, formulated in energetic terms, is based on the fundamental assumption that the surface energy density Φ depends in a non trivial manner on the displacement jump, unlike the Griffith model in which Φ is assumed to be constant. So in Dugdale's model, by assuming that the crack is always in mode I, *i.e.* that only the normal displacement is discontinuous, the surface energy density reads :

$$\Phi(\llbracket u_n \rrbracket) = \begin{cases} +\infty & \text{if} \quad \llbracket u_n \rrbracket < 0 \\ G_c\llbracket u_n \rrbracket/\delta_c & \text{if} \quad 0 \le \llbracket u_n \rrbracket \le \delta_c \\ G_c & \text{if} \quad \llbracket u_n \rrbracket \ge \delta_c \end{cases}. \tag{7}$$

In (7), $\llbracket u_n \rrbracket$ denotes the jump of the normal displacement, G_c is the critical energy release rate of Griffth's theory, whereas δ_c is an internal length characteristic of the cohesive forces model. The ratio G_c/δ_c has the dimension of a stress, say σ_c

$$\sigma_c = \frac{G_c}{\delta_c}. \tag{8}$$

In terms of the cohesive forces, the normal stress σ_{nn} giving the interaction between the crack lips is equal to σ_c as long as $0 < [\![u_n]\!] < \delta_c$ and vanishes as soon as $[\![u_n]\!] > \delta_c$,

$$\sigma_{nn} \begin{cases} \leq \sigma_c & \text{if} \quad [\![u_n]\!] = 0 \\ = \sigma_c & \text{if} \quad 0 < [\![u_n]\!] < \delta_c \\ = 0 & \text{if} \quad [\![u_n]\!] > \delta_c \end{cases} . \tag{9}$$

Therefore, the crack lips are generally divided into two zones: the so-called *cohesive zone* in which the cohesive forces are equal to σ_c and a so-called *non cohesive zone* in which there are no cohesive forces.

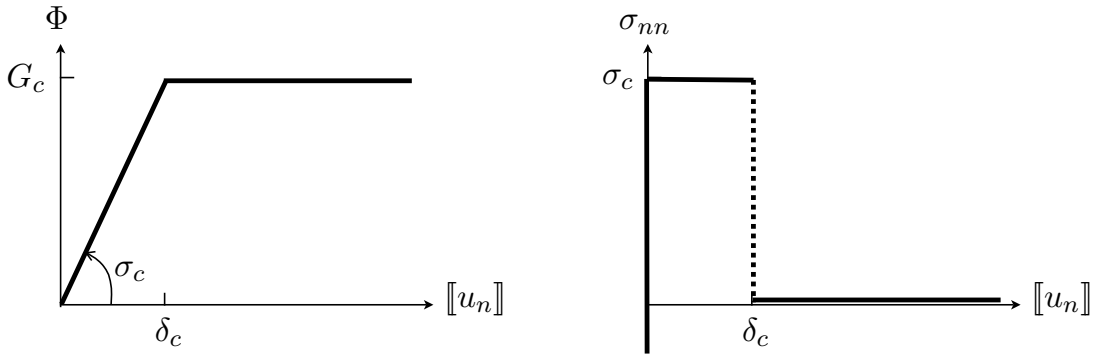


FIGURE 2. Surface energy density and cohesive forces in the model of Dugdale.

Remark 2. *The fact that σ_c plays also the role of the rupture stress for the material can be checked on the one-dimensional example of a bar under a simple traction. Indeed, if we follow [*Del Piero, 1999*], [*Charlotte et al., 2000*] or [*Bourdin et al., 2008*] and use a principle of energy minimization, it can be shown that the elastic response is no more a relative minimum of the total energy of the rod once the prescribed traction reaches the critical stress σ_c. In a full three-dimensional context, [*Charlotte et al., 2006*] show that the elastic response is a local minimum of the energy only if the stresses are less than σ_c* everywhere *in the body and thus that σ_c enters in the criterion of crack initiation. The direct consequence of the presence of a critical stress in the model is that a given structure can only sustain loads of limited amplitude.*

Remark 3. *The length δ_c characterizes the critical opening of a crack from which no more cohesive forces exist in Dugdale's model. It is a characteristic length of the material. But, in the plane strain calculations, another material characteristic length appears which involves also the elastic properties of the materials. This length is defined by*

$$d_c = \frac{\pi}{8(1-\nu^2)} \frac{E}{\sigma_c} \delta_c \tag{10}$$

and gives the order of magnitude of the length of the cohesive zone. In practice, since E is much greater than σ_c for usual materials, d_c is much greater than δ_c.

2.4. **General formulation of the crack evolution problem.** Owing to the symmetry and the smoothness assumptions above, we assume that a crack will nucleate at $(0,0)$ at the critical time t_e when the maximal tensile stress associated with the elastic response reaches the critical value σ_c, *i.e.*

$$t_e = \frac{\sigma_c}{\sigma_2^{\mathrm{el}}(0,0)}. \tag{11}$$

Then it is supposed that the crack will remain straight and along the axis $x_2 = 0$. Accordingly, the crack at time t, defined as the set of points where the displacement $\mathbf{u}(t)$ is discontinuous and denoted by $S_{\mathbf{u}(t)}$, is a subset of the part Γ of the axis $x_2 = 0$ included in Ω:

$$S_{\mathbf{u}(t)} = \{\mathbf{x} \in \Omega \ : \ [\![\mathbf{u}(t)]\!](\mathbf{x}) \neq 0\} \subset \Gamma = \Omega \cap \{x_2 = 0\}.$$

Moreover, still by symmetry, the jump of the displacement across the lips of the crack will be assumed to be purely normal. So, the crack is in mode I and the normal jump $[\![u_2(t)]\!](x_1)$ of the displacement at $(x_1, 0) \in S_{\mathbf{u}(t)}$ is called the *opening* of the crack at x_1.

Under this assumption on the crack path, the problem giving the displacement field at time t and hence the crack state at that time can be formulated by using a variational approach, like in [Ferdjani et al., 2007; Bourdin et al., 2008; Ferdjani et al., 2009]. Specifically, let $\mathcal{U}_{ad}(t)$ be the set of kinematically admissible displacement fields at time t, *i.e.* the set of smooth vector fields which satisfy the kinematic boundary conditions, which are allowed to jump on Γ only and whose normal jump is non negative:

$$\mathcal{U}_{ad}(t) = \{\mathbf{u}^* \in H^1(\Omega \setminus \Gamma; \mathbb{R}^2) \ : \ \mathbf{u}^* = t\mathbf{U} \text{ on } \partial_D\Omega, \quad [\![u_2^*]\!] \geq 0 \text{ on } \Gamma\} \tag{12}$$

where H^1 denotes the usual Sobolev space equipped with its natural norm $\left\| \cdot \right\|_1$. For a given t, at each $\mathbf{u}^* \in \mathcal{U}_{ad}(t)$ is associated the total energy of the body $\mathcal{E}_t(\mathbf{u}^*)$ as the sum of its elastic energy, its surface energy and the potential of the applied forces. Namely, $\mathcal{E}_t(\mathbf{u}^*)$ reads as

$$\mathcal{E}_t(\mathbf{u}^*) = \int_{\Omega\setminus\Gamma} \left(\frac{\lambda}{2} (\text{tr } \boldsymbol{\varepsilon}(\mathbf{u}^*))^2 + \mu \boldsymbol{\varepsilon}(\mathbf{u}^*) \cdot \boldsymbol{\varepsilon}(\mathbf{u}^*) \right) dx + \int_\Gamma \Phi([\![u_2^*]\!]) \, dx_1 - \int_{\partial_N\Omega} t\mathbf{F} \cdot \mathbf{u}^* \, dS. \tag{13}$$

We are now in a position to give a precise formulation of the crack evolution problem. That leads to the following

Definition 1 (Variational Formulation of the crack evolution problem)**.** *At each time $t \geq 0$, the displacement field $\mathbf{u}(t)$ is a local minimizer of the total energy $\mathcal{E}_t$ among the set of all kinematically admissible displacement fields $\mathcal{U}_{ad}(t)$. Specifically, $\mathbf{u}(t)$ must be such that*

(1) $\mathbf{u}(t) \in \mathcal{U}_{ad}(t)$;
(2) $\exists h(t) > 0$ such that $\mathcal{E}_t(\mathbf{u}(t)) \leq \mathcal{E}_t(\mathbf{u}^*)$, $\forall \mathbf{u}^* \in \mathcal{U}_{ad}(t)$ such that $\left\|\mathbf{u}^* - \mathbf{u}(t)\right\|_1 \leq h(t)$.

The stress field at time t is given by $\boldsymbol{\sigma}(t) = \lambda tr\boldsymbol{\varepsilon}(\mathbf{u}(t))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}(t))$, whereas the crack at time t corresponds to the jump set $S_{\mathbf{u}(t)}$.

Remark 4. *(Various advantages of a variational formulation) The variational formulation of the crack evolution problem presents several benefits by comparison with other formulations only based on equilibrium equations and constitutive conditions, namely*

(1) That allows to formulate the problem in a condensed form which remains valid even if one changes the behavior, the loading or the geometry;
(2) It contains in a unique formulation both the concepts of equilibrium and stability. Indeed, it turns out that the classical formulations based on equilibrium equations and constitutive conditions are simply first order stability conditions, as it is proved in Proposition 1;
*(3) The variational formulation supplies natural numerical methods to construct approximate solutions, see [*Lorentz, 2008*].*

Remark 5 (Absence of irreversibility conditions)**.** *Let us emphasize that no irreversibility conditions have been introduced in Dugdale's model and hence in the evolution problem. The absence of an explicit irreversibility condition allows us to simplify the presentation, but can be seen as a weakness of the formulation, because that could lead to unphysical responses. We will discuss this point when we will construct a solution in the next sections. The reader interested by a complete formulation taking account of the irreversibility should refer to [*Jaubert and Marigo, 2006; Bourdin et al., 2008; Abdelmoula et al., 2010*] where the issue of fatigue modeling is considered.*

Let us now establish the local conditions that $\mathbf{u}(t)$ and $\boldsymbol{\sigma}(t)$ must satisfy to be a solution of the variational problem stated in Definition 1.

Proposition 1 (First order stability conditions)**.** *The displacement field* $\mathbf{u}(t)$ *and its associated stress field* $\boldsymbol{\sigma}(t)$ *are solutions of the variational problem of Definition 1 only if they satisfy the following local conditions:*

$$\textit{Equilibrium equations} \quad : \quad \operatorname{div}\boldsymbol{\sigma}(t) = \mathbf{0} \text{ in } \Omega \setminus \Gamma \tag{14}$$

$$\textit{Boundary conditions} \quad : \quad \mathbf{u}(t) = t\mathbf{U} \text{ on } \partial_D\Omega, \quad \boldsymbol{\sigma}(t)\mathbf{n} = t\mathbf{F} \text{ on } \partial_N\Omega \tag{15}$$

$$\textit{Crack path conditions} \quad : \quad \begin{cases} [\![u(t)_2]\!] = 0, \quad \sigma(t)_{12} = 0, \quad \sigma(t)_{22} \le \sigma_c := \Phi'(0+) & \text{on } \Gamma \setminus S_{\mathbf{u}(t)} \\ [\![u(t)_2]\!] \ge 0, \quad \sigma(t)_{12} = 0, \quad \sigma(t)_{22} = \Phi'([\![u(t)_2]\!]) & \text{on } S_{\mathbf{u}(t)} \end{cases} \tag{16}$$

Proof. We only give a sketch of the proof which is based on classical variational arguments. However, the proof is valid for any cohesive force model and not merely for Dugdale's model. The method consists in considering kinematically admissible displacement fields of the form $\mathbf{u}^* = \mathbf{u}(t) + h\mathbf{v}$ with $h > 0$ sufficiently small. Inserting such a $\mathbf{u}^*$ into the stability condition gives $\mathcal{E}_t(\mathbf{u}(t)) \le \mathcal{E}_t(\mathbf{u}(t) + h\mathbf{v})$. Then dividing by h and passing to the limit when $h \to 0$ lead to

$$\int_{\Omega\setminus\Gamma} \boldsymbol{\sigma}(t) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx + \int_\Gamma \Phi'([\![u(t)_2]\!])[\![v_2]\!] \, dx_1 - \int_{\partial_N\Omega} t\mathbf{F} \cdot \mathbf{v} \, dS \ge 0. \tag{17}$$

Let us consider different types of fields $\mathbf{v}$.

(1) Let $\mathbf{v}$ be a smooth field such that $[\![\mathbf{v}]\!] = \mathbf{0}$ on Γ and $\mathbf{v} = \mathbf{0}$ on $\partial_D\Omega$. Then by classical arguments one deduces that

$$\begin{cases} \operatorname{div}\boldsymbol{\sigma}(t) = 0 & \text{in } \Omega \setminus \Gamma \\ \boldsymbol{\sigma}(t)\mathbf{n} = t\mathbf{F} & \text{on } \partial_N\Omega \\ [\![\boldsymbol{\sigma}(t)]\!]\mathbf{e}_2 = \mathbf{0} & \text{on } \Gamma \end{cases} \tag{18}$$

(2) After an integration by parts of the first integral of (17) and using (18), we obtain the following inequality

$$\int_\Gamma \Big(\Phi'([\![u(t)_2]\!]) - \sigma(t)_{22}\Big)[\![v_2]\!] \, dx_1 - \int_\Gamma \sigma(t)_{12}[\![v_1]\!] \, dx_1 \ge 0 \tag{19}$$

which must hold for any admissible $\mathbf{v}$.

(3) Since $[\![v_1]\!]$ can be chosen arbitrarily on Γ, one gets $\sigma(t)_{12} = 0$ on Γ. Then, after inserting this latter equality in (19) and dividing Γ into $S_{\mathbf{u}(t)}$ and $\Gamma \setminus S_{\mathbf{u}(t)}$, (19) becomes

$$\int_{S_{\mathbf{u}(t)}} \Big(\Phi'([\![u(t)_2]\!]) - \sigma(t)_{22}\Big)[\![v_2]\!] \, dx_1 + \int_{\Gamma\setminus S_{\mathbf{u}(t)}} \Big(\sigma_c - \sigma(t)_{22}\Big)[\![v_2]\!] \, dx_1 \ge 0, \tag{20}$$

where $\sigma_c := \Phi'(0+)$. Finally, since $[\![v_2]\!]$ can be chosen arbitrarily on $S_{\mathbf{u}(t)}$ whereas $[\![v_2]\!]$ is necessarily non negative on $\Gamma \setminus S_{\mathbf{u}(t)}$ in order that $[\![u(t)_2 + hv_2]\!] \ge 0$ on Γ, one gets

$$\begin{cases} \sigma(t)_{22} \le \sigma_c & \text{on } \Gamma \setminus S_{\mathbf{u}(t)} \\ \sigma(t)_{22} = \Phi'([\![u(t)_2]\!]) & \text{on } S_{\mathbf{u}(t)} \end{cases} \tag{21}$$

The proof is complete. □

Remark 6. *Let us emphasize the most important results contained in Proposition 1.*

- *The equilibrium equations, boundary conditions and crack path conditions are only* necessary *conditions in order that* $\mathbf{u}(t)$ *be stable in the sense of Definition 1. In general, they are not sufficient and one must add* second order *stability conditions. However, in the present paper, we will not introduce these second order stability conditions and the interested reader should refer to [*Charlotte et al., 2000; Bourdin et al., 2008*] for more details on their use.*

- *The crack path conditions contain not only the fact that the cohesive forces on the lips of the crack are given by the derivative of the surface energy density, namely* $\sigma(t)_{22} = \Phi'([\![u(t)_2]\!])$*, but also the fact that* $\sigma_c = \Phi'(0+)$ *plays the role of a yield stress criterion for the nucleation of a crack. Indeed, one must have* $\sigma(t)_{22} \leq \sigma_c$ *everywhere (on* Γ*). Accordingly, the stress field is necessarily bounded and no singularity is allowed.*

3. Resolution of the crack evolution problem

3.1. **Reduction of the problem with the help of the symmetry assumptions.** The evolution problem which is stated above in its general form can be reduced with the help of the symmetry assumptions introduced in Subsection 2.2. Let us note however that, since the uniqueness of the solution is not ensured, the search for a solution respecting these symmetries constitutes additional assumptions. The first extra assumption is the following:

Hypothesis 1 (Centered crack path)**.** *At every time* $t > 0$*, the jump set* $S_{\mathbf{u}(t)}$ *is either empty or an interval centered at* $(0,0)$*, i.e. there exists* $a(t) \geq 0$ *such that* $S_{\mathbf{u}(t)} = (-a(t), a(t)) \times \{0\}$.

The second assumptions concerns the monotonicity of the opening on the crack lips.

Hypothesis 2 (Symmetry and monotonicity of $[\![u(t)_2]\!]$)**.** *When the jump set* $S_{\mathbf{u}(t)}$ *is not empty, the opening* $[\![u(t)_2]\!](x_1)$ *is an even continuous function of* x_1*, maximal at* $x_1 = 0$ *and decreasing to* 0 *when* $|x_1|$ *grows to* $a(t)$.

This second assumption limits the number of possibilities for the crack state at a given t. Specifically, we can distinguish the three following cases, see also Figure 3:

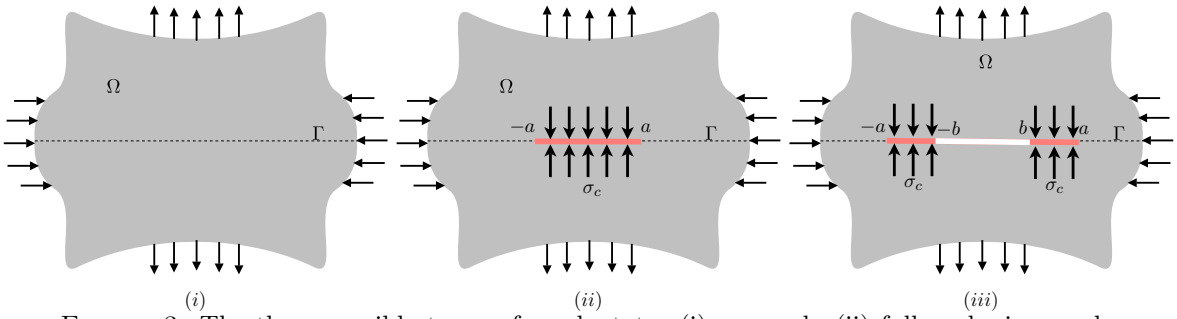


Figure 3. The three possible types of crack state: (i) no crack, (ii) fully cohesive crack, (iii) partially non cohesive crack

(*i*) there exists no crack, *i.e.* $a(t) = 0$ and $S_{\mathbf{u}(t)} = \emptyset$. The response is purely elastic and the set of all crack states of this type which satisfy the first order stability conditions is called the *elastic branch*;

(*ii*) a crack exists but its opening at $x_1 = 0$ is less than the critical value δ_c associated with Dugdale's model, *i.e.* $a(t) > 0$ and $[\![u(t)_2]\!](0) \leq \delta_c$. That corresponds to the case where the entire crack lips are submitted to the cohesive force σ_c. The set of all crack states of this type which satisfy the first order stability conditions is called the *fully cohesive branch*;

(*iii*) a crack exists and its opening at $x_1 = 0$ is greater than the critical value δ_c, *i.e.* $a(t) > 0$ and $[\![u(t)_2]\!](0) > \delta_c$. In that case, since the opening is a monotonic function of $|x_1|$ decreasing to 0, there exists two symmetrical points $(\pm b(t), 0)$ with $0 < b(t) < a(t)$ where the opening is equal to δ_c. Therefore, by virtue of Dugdale's model, the crack is divided into two parts:
   (a) the *cohesive zone* where the cohesive forces are equal to σ_c;
   (b) the *non cohesive zone* where the cohesive forces vanish.

Specifically, that corresponds to the case where $a(t) > b(t) > 0$ and

$$\begin{cases} \text{if } |x_1| < b(t), & \text{then } [\![u(t)_2]\!](x_1) > \delta_c, \quad \sigma(t)_{22}(x_1, 0) = 0 \\ \text{if } b(t) < |x_1| < a(t), & \text{then } 0 < [\![u(t)_2]\!](x_1) < \delta_c, \quad \sigma(t)_{22}(x_1, 0) = \sigma_c \end{cases}.$$

The set of all crack states of this type which satisfy the first order stability conditions is called the *partially non cohesive branch*.

Of course the positions $a(t)$ and $b(t)$ of the tips of the cohesive zone and non cohesive zone, when they exist, have to be determined. This is the absence of singularity which supplies the equation giving $a(t)$, as it is shown in the following Proposition:

Proposition 2 (Vanishing of the stress intensity factor $\mathsf{K_I}$ at the tips of the cohesive crack)**.** *Since the normal stress $\sigma(t)_{22}$ must be bounded everywhere on Γ, no singularity can exist at the tips of the cohesive crack and hence the stress intensity factor $\mathsf{K_I}$ must vanish at $x_1 = \pm a(t)$.*

Proof. Let us consider the case where a crack exists, *i.e.* $a(t) > 0$. Then, the displacement field $\mathbf{u}(t)$ and the stress field $\boldsymbol{\sigma}(t)$ must satisfy in the neighborhood of the crack tips $(\pm a(t), 0)$ the elasticity equations with the boundary conditions $\boldsymbol{\sigma}(t)\mathbf{e}_2 = \sigma_c \mathbf{e}_2$ on the lips of the crack (close to the tips). Therefore, we are in the situation of a crack in a linear elastic isotropic medium submitted to Neumann boundary conditions. The structure of the solution is then well known, cf [Bui, 1978; Grisvard, 1985], and contains *a priori* a singular part. Specifically, by virtue of the symmetry assumptions, the crack is in mode I and the displacement $\mathbf{u}(t)$ in the neighborhood of the tip $(a(t), 0)$ reads as

$$\mathbf{u}(t)(\mathbf{x}) = \frac{\mathsf{K_I}}{2\mu}\sqrt{\frac{r}{2\pi}}(3 - 4\nu - \cos\theta)\left(\cos\frac{\theta}{2}\mathbf{e}_1 + \sin\frac{\theta}{2}\mathbf{e}_2\right) + \text{regular terms},$$

where (r, θ) denotes the polar coordinates of $\mathbf{x}$, *i.e.* $\mathbf{x} = r\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$. Accordingly, the normal jump of the displacement on the lips of the crack near $(a(t), 0)$ is given by

$$[\![u(t)_2]\!](x_1) = 8(1 - \nu^2)\frac{\mathsf{K_I}}{E}\sqrt{\frac{a(t) - x_1}{2\pi}} + \text{regular terms}.$$

Therefore, one must have $\mathsf{K_I} \geq 0$ in order that $[\![u(t)_2]\!](x_1) \geq 0$. On the other hand, the normal stress field $\sigma(t)_{22}$ on the axis $\theta = 0$ near the tip $(a(t), 0)$ reads as

$$\sigma(t)_{22}(x_1) = \frac{\mathsf{K_I}}{\sqrt{2\pi(x_1 - a(t))}} + \text{regular terms}.$$

But since $\sigma(t)_{22}(x_1)$ must be no greater than σ_c, one must have also $\mathsf{K_I} \leq 0$ and hence finally $\mathsf{K_I} = 0$. □

As far as the position $b(t)$ of the non cohesive crack tips is concerned, we simply have by construction

Proposition 3 (Critical opening at the non cohesive crack tip)**.** *The position $b(t)$ of the non cohesive crack tips, when they exist, must be such that the opening at these tips be equal to δ_c:*

$$[\![u(t)_2]\!](\pm b(t)) = \delta_c. \tag{22}$$

We are now in a position to exhibit a method for constructing a solution of the crack evolution problem. Specifically, the procedure is the following one:

(*i*) One solves the elastic problem and determine the fields $(\mathbf{u}^{\text{el}}, \boldsymbol{\sigma}^{\text{el}})$. One deduces the elastic branch which corresponds to $\mathbf{u}(t) = t\mathbf{u}^{\text{el}}$ for $0 \leq t \leq t_e$. (Indeed, for $t > t_e$, the elastic response $t\mathbf{u}^{\text{el}}$ cannot be a solution of the crack evolution problem, because $t\sigma_{22}^{el} > \sigma_c$ somewhere on Γ by virtue of the definition of t_e.)

(*ii*) One considers the case of a fully cohesive crack with a length $2a > 0$ at time $t > 0$. For given a and t, we define the associated displacement and stress fields as the *unique* solution, denoted $(\mathbf{u}[t,a], \boldsymbol{\sigma}[t,a])$, of the following linear elastic problem posed on the cracked body with uniform cohesive forces on the crack lips:

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma}[t,a] = \mathbf{0} & \text{in} \quad \Omega \setminus ([-a,a] \times \{0\}) \\ \boldsymbol{\sigma}[t,a] = \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}[t,a]))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}[t,a]) & \text{in} \quad \Omega \setminus ([-a,a] \times \{0\}) \\ \mathbf{u}[t,a] = t\mathbf{U} & \text{on} \quad \partial_D\Omega, \\ \boldsymbol{\sigma}[t,a]\mathbf{n} = t\mathbf{F} & \text{on} \quad \partial_N\Omega \\ \boldsymbol{\sigma}[t,a]\mathbf{e}_2 = \sigma_c\mathbf{e}_2 & \text{on} \quad [-a,a] \times \{0\} \end{cases} \tag{23}$$

Note that this problem admits the same symmetries as the original elastic problem and hence its solution too.

From its solution, one deduces the value of the mode I stress intensity factor $\mathsf{K}_\mathsf{I}[t,a]$ at the tips of the cohesive crack. Requiring that it vanishes, one obtains the right value $a(t)$ of the position of the crack tip:

$$\mathsf{K}_\mathsf{I}[t,a(t)] = 0. \tag{24}$$

Of course, it is not ensured at this stage that there exists a unique solution of (24) for $a(t)$. If several solutions exist, then one can define several fully cohesive branches, but one can expect than only one starts from $a = 0$ at $t = t_e$.

Then, one defines $(\mathbf{u}(t), \boldsymbol{\sigma}(t))$ by $(\mathbf{u}[t,a(t)], \boldsymbol{\sigma}[t,a(t)])$ and one must find for what values of t $(\mathbf{u}(t), \boldsymbol{\sigma}(t))$ satisfies the first order stability conditions. The main condition is that the opening must be positive and less than δ_c everywhere on the crack lips:

$$0 < [\![u(t)_2]\!] < \delta_c \quad \text{on} \quad (-a(t), a(t)). \tag{25}$$

If this condition is satisfied only when t lies in some interval (t_e, t_i), then this interval will constitute the fully cohesive branch. Finally, it will remain to check that $\sigma(t)_{22} \leq \sigma_c$ everywhere on Γ for those values of t.

(*iii*) One considers finally the case of a partially non cohesive crack at time t whose non cohesive length is $2b$ and the tips of the cohesive zones are at $\pm a$. For given (a,b,t) with $0 < b < a$ and $t > 0$, we define the associated displacement and stress fields as the *unique* solution, denoted $(\mathbf{u}[t,a,b], \boldsymbol{\sigma}[t,a,b])$, of the following linear elastic problem posed on the cracked body with non uniform cohesive forces on the crack lips:

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma}[t,a,b] = \mathbf{0} & \text{in} \quad \Omega \setminus ([-a,a] \times \{0\}) \\ \boldsymbol{\sigma}[t,a,b] = \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u}[t,a,b]))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}[t,a,b]) & \text{in} \quad \Omega \setminus ([-a,a] \times \{0\}) \\ \mathbf{u}[t,a,b] = t\mathbf{U} & \text{on} \quad \partial_D\Omega, \\ \boldsymbol{\sigma}[t,a,b]\mathbf{n} = t\mathbf{F} & \text{on} \quad \partial_N\Omega \\ \boldsymbol{\sigma}[t,a,b]\mathbf{e}_2 = \mathbf{0} & \text{on} \quad (-b,b) \times \{0\} \\ \boldsymbol{\sigma}[t,a,b]\mathbf{e}_2 = \sigma_c\mathbf{e}_2 & \text{on} \quad ((-a,-b) \cup (b,a)) \times \{0\} \end{cases} \tag{26}$$

This problem also admits the same symmetries as the original elastic problem and hence its solution too.

From the solution, one deduces the value of the mode I stress intensity factor $\mathsf{K}_\mathsf{I}[t,a,b]$ at the tips of the cohesive crack. Requiring that it vanishes, one obtains a first equation for the right values $a(t)$ and $b(t)$ of the crack tips position at time t. The second equation is given by the opening at

$\pm b(t)$. Accordingly, the system of equations for $(a(t), b(t))$ reads as

$$\begin{cases} \mathsf{K_I}[t, a(t), b(t)] = 0 \\ [\![u[t, a(t), b(t)]_2]\!](b(t)) = \delta_c \end{cases} . \tag{27}$$

We are ensured neither that a solution of (27) for $(a(t), b(t))$ exists, nor that the solution is unique. We can expect that a solution exists only for some values of t. If several solutions exist, then one can define several partially non cohesive branches, but one can expect than only one starts from $a = a_i$ and $b = 0$ at $t = t_i$.

Then, for any solution $(a(t), b(t))$ at time t, one defines the displacement and stress fields $(\mathbf{u}(t), \boldsymbol{\sigma}(t))$ by $(\mathbf{u}[t, a(t), b(t)], \boldsymbol{\sigma}[t, a(t), b(t)])$. It remains to check that $(\mathbf{u}(t), \boldsymbol{\sigma}(t))$ really satisfies all the first order stability conditions. Specifically, it remains only to verify that

$$\begin{cases} [\![u(t)_2]\!](x_1) > \delta_c & \text{if} \quad |x_1| < b(t) \\ 0 < [\![u(t)_2]\!](x_1) < \delta_c & \text{if} \quad b(t) < |x_1| < a(t) \\ \sigma(t)_{22}(x_1) \leq \sigma_c & \text{if} \quad |x_1| \geq a(t) \end{cases} . \tag{28}$$

3.2. **Analytical calculation with a two-scale approach.** In this section we construct a solution of the crack evolution problem in a closed form, but under the condition that the material characteristic length d_c defined in (10) is small by comparison with the characteristic length L of the body. Moreover, we assume that the stress field is genuinely non uniform by considering that ℓ is of the same order as or much smaller than L :

$$d_c \ll L, \qquad \ell \lesssim L.$$

This hierarchy of the lengths allows us to use a two-scale approach to construct the solution (which will be hence an approximate solution only). The construction follows the procedure described in the previous subsection.

3.2.1. *Determination of the elastic branch.* Once the original elastic problem (3) is solved, one obtains the normal stress distribution $\sigma_{22}^{\mathrm{el}}(x_1)$ along the axis Γ and therefore by (11) the time t_e which corresponds to the limit of the validity of the elastic response.

3.2.2. *Determination of the fully cohesive branch.* Let $t > 0$ and a such that $0 < a \ll L$. In such a case, since the crack length is small, the crack should perturb the elastic fields only in a neighborhood of the origin. Therefore, if we introduce in (23) the gaps of the solution with the elastic fields, *i.e.*

$$\bar{\mathbf{u}}[t, a] = \mathbf{u}[t, a] - t\mathbf{u}^{\mathrm{el}}, \qquad \bar{\boldsymbol{\sigma}}[t, a] = \boldsymbol{\sigma}[t, a] - t\boldsymbol{\sigma}^{\mathrm{el}},$$

then $\bar{\boldsymbol{\sigma}}[t, a](\mathbf{x})$ should tend to 0 when $\|\mathbf{x}\|$ becomes large by comparison with a. Moreover, on the lips of the crack the gap of the normal stress verifies $\bar{\sigma}[t, a]_{22}(x_1) = \sigma_c - t\Sigma(x_1)$ where $\Sigma(x_1)$ is given by (6). Accordingly, using (11) leads to

$$\bar{\sigma}[t, a]_{22}(x_1) = \left(1 - \frac{t}{t_e}\right)\sigma_c + 2\frac{t}{t_e}\frac{x_1^2}{\ell^2}\sigma_c + o(x_1^2), \quad |x_1| < a.$$

When $a \ll \ell$ we can neglect the term $o(x_1^2)$ and only consider the first two terms of the expansion. But even if a is of the same order as ℓ, one can consider that the parabolic distribution of the normal stress is the simplest case to study the influence of the stress gradient on the nucleation of a crack. These considerations

allow us to write the problem giving the gaps in the neighborhood of the origin as follows

$$\begin{cases} \operatorname{div} \bar{\boldsymbol{\sigma}}[t,a] = \mathbf{0} & \text{in} \quad \mathbb{R}^2 \setminus ([-a,a]\times\{0\}) \\ \bar{\boldsymbol{\sigma}}[t,a] = \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\bar{\mathbf{u}}[t,a]))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\bar{\mathbf{u}}[t,a]) & \text{in} \quad \mathbb{R}^2 \setminus ([-a,a]\times\{0\}) \\ \bar{\boldsymbol{\sigma}}[t,a](\mathbf{x}) \to \mathbf{0} & \text{when} \quad |\mathbf{x}| \to \infty, \\ \bar{\boldsymbol{\sigma}}[t,a](x_1,0)\mathbf{e}_2 = \left(1 - \dfrac{t}{t_e} + 2\dfrac{t}{t_e}\dfrac{x_1^2}{\ell^2}\right)\sigma_c \mathbf{e}_2 & \text{when} \quad x_1 \in (-a,a) \end{cases} \tag{29}$$

Thus the construction of this simplified problem is based on an approximation and an assumption:

(1) The problem is posed on the whole plane $\mathbb{R}^2$ with the condition that the stresses must decrease to 0 at infinity. We use the fact that a is small by comparison to L and hence that we must recover the elastic solution far enough from the crack. The approximation simply consists in treating the problem at the scale of a and in sending to infinity the boundary of the body;
(2) The normal stress repartition on the crack lips is assumed to be quadratic. This can be considered either as a particular case of loading or an approximation valid when a is small by comparison to ℓ.

Owing to these approximations it becomes possible to solve the problem (29) in a closed form. Indeed the simplified problem is a particular case of a family of plane elastic problems which can be solved with the method of complex potentials developed by [Muskhelishvili, 1963]. The main steps of the method are recalled in the appendix and we can directly use the results by identifying the normal stress distribution with

$$\mathtt{T}(x_1) = \left(1 - \frac{t}{t_e} + 2\frac{t}{t_e}\frac{x_1^2}{\ell^2}\right)\sigma_c.$$

The stress intensity factor $\mathsf{K}_\mathsf{I}[t,a]$ is given by (53) and after the calculation of the integral one gets

$$\mathsf{K}_\mathsf{I}[t,a] = \sigma_c\sqrt{\pi a}\left(\left(1 - \frac{a^2}{\ell^2}\right)\frac{t}{t_e} - 1\right).$$

The condition $\mathsf{K}_\mathsf{I}[t,a] = 0$ gives the position of the crack tips in function of time:

$$a(t) = \ell\sqrt{1 - \frac{t_e}{t}}. \tag{30}$$

Hence the crack length is proportional to the characteristic stress gradient length, starts from 0 at t_e and then increases with time. This solution is valid as long as the opening of the crack at $x_1 = 0$ remains less than δ_c, and provided that the normal stress is less that σ_c all along the axis.

The normal stress and the opening are obtained by integration. First, from (51) one gets

$$\varphi'(z) = \frac{\sigma_c}{2}\left(\frac{t}{t_e} - 1\right)\left(\frac{2z^2}{a(t)^2} - 1 - \frac{2z}{a(t)}\sqrt{\frac{z^2}{a(t)^2} - 1}\right) \tag{31}$$

where $a(t)$ is given by (30). Therefore, by virtue of (49), one obtains

$$\bar{\sigma}[t,a(t)]_{22}(x_1,0) = \sigma_c\left(\frac{t}{t_e} - 1\right)\left(\frac{2x_1^2}{a(t)^2} - 1 - \frac{2|x_1|}{a(t)}\sqrt{\frac{x_1^2}{a(t)^2} - 1}\right) \quad \text{if} \quad |x_1| \geq a(t).$$

One deduces that the $\bar{\sigma}[t,a(t)]_{22}(x_1,0)$ is a monotonically decreasing function of $|x_1|$, decreasing from $\sigma_c(t/t_e - 1)$ at the crack tips to 0 at infinity. It is then easy to check that the normal stress $\sigma(t)_{22}(x_1,0)$ is less than σ_c for all x_1.

Remark 7. *The above expressions of the normal stress are based on the assumption that $a(t) \ll L$ and are only valid at a small scale. With these approximations we can simply conclude that the normal stress is less than σ_c at small scale, i.e. in a neighborhood of the origin. It could happen that the maximal traction criterion be reached at another point (far from the origin) at time t. In such a case, another crack would*

nucleate at that point. But in the first stage of their growth, the cracks do not interact each other and the present procedure remains valid.

Let us now determine the opening of the crack. Let $z^{\pm} = x_1 \pm \mathtt{i}0$ be the points on the lips of the crack at x_1, $|x_1| \leq a(t)$. It comes from (31) that

$$\varphi'(z^{\pm}) = \frac{\sigma_c}{2}\left(\frac{t}{t_e} - 1\right)\left(\frac{2x_1^2}{a(t)^2} - 1 \mp \mathtt{i}\frac{2x_1}{a(t)}\sqrt{1 - \frac{x_1^2}{a(t)^2}}\right)$$

and hence that

$$[\![\varphi]\!]'(x_1) = -2\mathtt{i}\sigma_c\left(\frac{t}{t_e} - 1\right)\frac{x_1\sqrt{a(t)^2 - x_1^2}}{a(t)^2}.$$

By a straightforward integration and using the fact that $[\![\varphi]\!](\pm a(t)) = 0$, one gets $[\![\varphi]\!](x_1)$. Finally, by virtue of (50), the opening reads as

$$[\![u(t)_2]\!](x_1) = \frac{8}{3}(1-\nu^2)\frac{\sigma_c}{E}a(t)\left(\frac{t}{t_e} - 1\right)\left(1 - \frac{x_1^2}{a(t)^2}\right)^{3/2}. \tag{32}$$

Let us remark that the opening is an even function of x_1, maximal at the origin which justifies *a posteriori* **Hypothesis 2**. Using (30) and the definition (10) of d_c, the opening at the origin can read as

$$[\![u(t)_2]\!](0) = \frac{\pi\ell}{3d_c}\delta_c\sqrt{\frac{t_e}{t}}\left(\frac{t}{t_e} - 1\right)^{3/2}. \tag{33}$$

Hence the opening at $x_1 = 0$ is a monotonically increasing function of t for $t \geq t_e$. It will reach the critical value δ_c at time t_i given by

$$\frac{t_e}{t_i}\left(\frac{t_i}{t_e} - 1\right)^3 = \frac{9d_c^2}{\pi^2\ell^2}. \tag{34}$$

The time t_i corresponds to the end of the fully cohesive branch, after which a non cohesive zone will appear at the center of the crack. At that time, the half-length of the cohesive crack is $a_i = a(t_i)$.

When d_c is much smaller than ℓ, t_i can be approximated by

$$\frac{t_i}{t_e} \approx 1 + \left(\frac{3d_c}{\pi\ell}\right)^{2/3}. \tag{35}$$

and the half-length of the crack at time t_i is given by

$$a_i \approx \left(\frac{3d_c}{\pi\ell}\right)^{1/3}\ell.$$

Let us note that the order of magnitude of a_i is intermediate between d_c and ℓ:

$$d_c \ll a_i \ll \ell.$$

3.2.3. *Determination of the partially non cohesive branch.* Let us now consider, at a given time $t > 0$, the case of a partially non cohesive crack whose non cohesive length is $2b$ whereas the tips of the cohesive zones are at $\pm a$. We assume that $0 < b < a \ll L$ and hence, still, that the crack perturbs the elastic fields in a neighborhood of the origin only. Therefore, introducing in (26) the gaps of the solution with the elastic fields, *i.e.*

$$\bar{\mathbf{u}}[t,a,b] = \mathbf{u}[t,a,b] - t\mathbf{u}^{\mathrm{el}}, \qquad \bar{\boldsymbol{\sigma}}[t,a,b] = \boldsymbol{\sigma}[t,a,b] - t\boldsymbol{\sigma}^{\mathrm{el}},$$

and using the same approximations as in the case of a fully cohesive crack, the local problem reads as follows

$$\begin{cases} \operatorname{div} \bar{\boldsymbol{\sigma}}[t,a,b] = \mathbf{0} & \text{in} \quad \mathbb{R}^2 \setminus ([-a,a]\times\{0\}) \\ \bar{\boldsymbol{\sigma}}[t,a,b] = \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\bar{\mathbf{u}}[t,a,b]))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\bar{\mathbf{u}}[t,a,b]) & \text{in} \quad \mathbb{R}^2 \setminus ([-a,a]\times\{0\}) \\ \bar{\boldsymbol{\sigma}}[t,a,b](\mathbf{x}) \to \mathbf{0} & \text{when} \quad |\mathbf{x}| \to \infty \\ \bar{\boldsymbol{\sigma}}[t,a,b](x_1,0)\mathbf{e}_2 = \mathtt{T}(x_1)\mathbf{e}_2 & \text{on} \quad (-a,a)\times\{0\} \end{cases} \tag{36}$$

where

$$\mathtt{T}(x_1) = \begin{cases} \left(-\dfrac{t}{t_e} + 2\dfrac{t}{t_e}\dfrac{x_1^2}{\ell^2}\right)\sigma_c & \text{if} \quad |x_1| < b \\ \left(1 - \dfrac{t}{t_e} + 2\dfrac{t}{t_e}\dfrac{x_1^2}{\ell^2}\right)\sigma_c & \text{if} \quad b < |x_1| < a \end{cases} \tag{37}$$

The problem (36) can be solved in a closed form, because it is still a particular case of the family of plane elastic problems presented in the appendix. For a given triple (t,a,b), the solution $\mathbf{u}[t,a,b]$ is unique up to a rigid displacement field and the stress field $\boldsymbol{\sigma}[t,a,b]$ is unique. For a given $t>0$, the solution belongs to the partially non cohesive branch only if a and b are such that

$$\begin{cases} \mathsf{K}_{\mathsf{I}}[t,a,b] = 0 \\ [\![u[t,a,b]_2]\!](b) = \delta_c \end{cases} \tag{38}$$

The stress intensity factor $\mathsf{K}_{\mathsf{I}}[t,a,b]$ is given by (53) and after the calculation of the integral one gets

$$\mathsf{K}_{\mathsf{I}}[t,a,b] = \sigma_c\sqrt{\pi a}\left(\left(1-\frac{a^2}{\ell^2}\right)\frac{t}{t_e} - 1 + \frac{2}{\pi}\arcsin\frac{b}{a}\right). \tag{39}$$

The vanishing condition of the stress intensity factor gives a first relation between a, b and t, namely

$$\left(1-\frac{a^2}{\ell^2}\right)\frac{t}{t_e} = \frac{2}{\pi}\arccos\frac{b}{a}. \tag{40}$$

The calculation of the opening $[\![u[t,a,b]_2]\!](b)$ requires to determine the jump of the complex potential $\varphi(z)$ on the lips of the cohesive crack, *i.e.* $[\![\varphi]\!](x_1)$ for $b<|x_1|<a$. Using (37), (40) and (51), after a tedious calculation of the integral one eventually gets

$$[\![\varphi]\!]'(x_1) = -2\mathtt{i}\sigma_c\left(\frac{x_1\sqrt{a^2-x_1^2}}{\ell^2}\frac{t}{t_e} + \frac{1}{\pi}\operatorname{arctanh}\left(\frac{b\sqrt{a^2-x_1^2}}{x_1\sqrt{a^2-b^2}}\right)\right). \tag{41}$$

Since $[\![\varphi]\!](a)=0$, an integration of (41) leads to

$$[\![\varphi]\!](b) = 2\mathtt{i}\sigma_c\left(\frac{1}{3}\frac{t}{t_e}\frac{(a^2-b^2)^{3/2}}{\ell^2} + \frac{b}{\pi}\ln\frac{a}{b}\right)$$

and finally, by using (54), one obtains the opening at the tip of the non cohesive crack

$$[\![u[t,a,b]_2]\!](b) = 8(1-\nu^2)\frac{\sigma_c}{E}\left(\frac{1}{3}\frac{t}{t_e}\frac{(a^2-b^2)^{3/2}}{\ell^2} + \frac{b}{\pi}\ln\frac{a}{b}\right). \tag{42}$$

The requirement that this opening is equal to δ_c gives the second relation between a, b and t, specifically

$$\frac{\pi}{3}\frac{t}{t_e}\frac{(a^2-b^2)^{3/2}}{\ell^2} + b\ln\frac{a}{b} = d_c. \tag{43}$$

Remark 8. *It is possible to prove that if a triple (t,a,b) satisfies (40) and (43), then the associated fields $\mathbf{u}[t,a,b]$ and $\boldsymbol{\sigma}[t,a,b]$ satisfy the conditions (28) and hence verify all the first order stability conditions. The proof is based on a careful study of the complex potential $\varphi(z)$ for $z = x_1 \pm \mathtt{i}0$, but the calculations are too long to be reproduced here.*

3.2.4. *Representation of the three branches.*

(i) The elastic branch starts at $t = 0$ and finishes at $t = t_e$ given by (11). All along this branch, there is no crack, $a(t) = 0$. Therefore, the elastic branch corresponds to the segment line $[0, t_e] \times \{0\}$ in the (t, a) diagram.

(ii) The fully cohesive branch starts at $t = t_e$ and finishes at $t = t_i$ given by (35). Along this branch, the length $a(t)$ of the cohesive crack grows continuously with t from 0 to a_i, $a(t)$ being given by (30). Therefore, the fully cohesive branch corresponds to the monotonic curve represented on Figure 5 in the (t, a) diagram which starts from the point $(t_e, 0)$ and finishes at the point (t_i, a_i).

(iii) For the partially non cohesive branch, a, b and t are related by the two conditions (40) and (43). For studying these two conditions, let us set

$$\alpha = \frac{b}{a} \in (0, 1).$$

Then, using (40), (43) can read as

$$\left(\frac{2}{3}(1-\alpha^2)^{3/2}\arccos\alpha + \alpha\ln\alpha\right)\frac{a^3}{\ell^3} + \frac{d_c a^2}{\ell^3} - \alpha\ln\alpha\frac{a}{\ell} - \frac{d_c}{\ell} = 0. \tag{44}$$

For a given $\alpha \in (0, 1)$, (44) is a cubic equation for $\bar{a} := a/\ell$ which depends on the parameter $\epsilon := d_c/\ell$. It turns out that this equation admits a unique solution, say $\bar{a}^\epsilon(\alpha)$, whose dependence on α is non monotonic. Specifically, $\bar{a}^\epsilon(\alpha)$ starts from $a_i/\ell = \sqrt{1 - t_e/t_i}$ at $\alpha = 0$, then is first decreasing up to a_m/ℓ before to be increasing and finally tends to 1 when α tends to 1, cf Figure 4. Accordingly, (40) gives t/t_e as a function of α which depends also on ϵ, say $\bar{t}^\epsilon(\alpha)$:

$$\frac{t}{t_e} = \bar{t}^\epsilon(\alpha) := \frac{2\arccos\alpha}{\pi(1 - \bar{a}^\epsilon(\alpha)^2)}. \tag{45}$$

The function $\bar{t}^\epsilon(\alpha)$ starts from t_i/t_e at $\alpha = 0$ and is first monotonically decreasing up to t_l/t_e, that minimum being reached at α_l. Then, $\bar{t}^\epsilon(\alpha)$ grows to infinity when α grows to 1, cf Figure 4. Finally, the evolution of b with α is given by the function $\bar{b}^\epsilon(\alpha)$:

$$\frac{b}{\ell} = \bar{b}^\epsilon(\alpha) := \alpha\bar{a}^\epsilon(\alpha). \tag{46}$$

As it is shown on Figure 4, b is a monotonically increasing function of α, starting from 0 at $\alpha = 0$ and tending to ℓ when α tends to 1.

Accordingly, the triples (a, b, t) satisfying (40) and (43) can be seen as two parametric curves $(t(\alpha), a(\alpha))$ and $(t(\alpha), b(\alpha))$ parameterized by $\alpha \in (0, 1)$ and depending on ℓ and on the ratio d_c/ℓ. In particular the curve $(t(\alpha), a(\alpha))$ represents the partially non cohesive branch in the (t, a) diagram, cf Figure 5. Since the function $\bar{a}^\epsilon(\alpha)$ and $\bar{t}^\epsilon(\alpha)$ are non monotonic and monotonically decreasing for small α, the partially non cohesive branch contains a snap-back in the neighborhood of (t_i, a_i) and a limit point (t_l, a_l), both points depending on ℓ and d_c. Accordingly, the branch has the shape of a loop which can be divided into two parts: the lower part between (t_i, a_i) and (t_l, a_l), the upper part after (t_l, a_l).

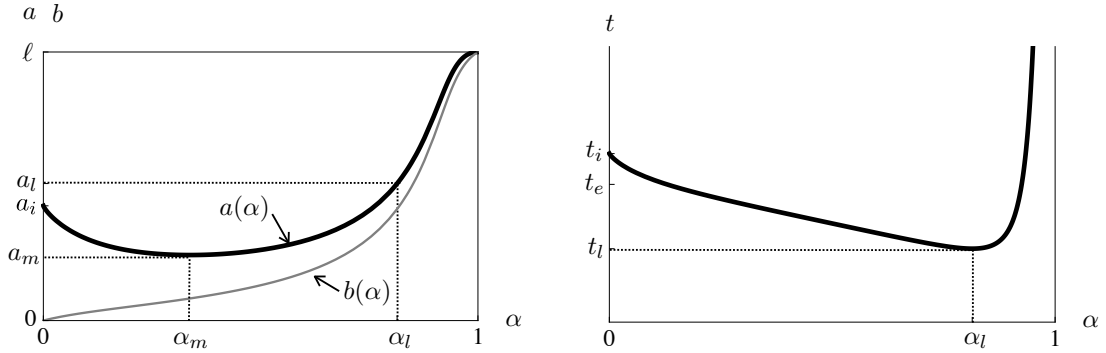


FIGURE 4. Typical dependence of a, b and t on $\alpha = b/a$. Here the curves correspond to the case where $d_c/\ell = 0.1$.

• Finally the three branches can be represented in a diagram (t, a) and one obtains typically the curves plotted in Figure 5. Their dependence on ℓ and d_c will be discussed in the next section.

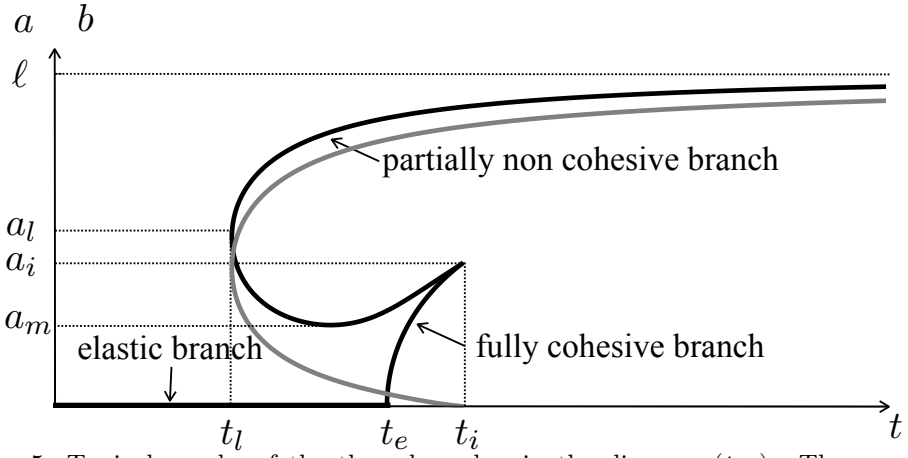


FIGURE 5. Typical graphs of the three branches in the diagram (t, a). The gray curve represents the evolution of the tip b of the non cohesive crack for the partially non cohesive branch. Here the curves correspond to the case where $d_c/\ell = 0.1$.

Remark 9. *The fact that a and b tends to a limit, namely ℓ, when t tends to infinity is due to the fact that the elastic response leads to a negative normal stress distribution at large distance of the origin, see Remark 1. Note however that the limit ℓ is greater than the distance $\ell/\sqrt{2}$ at which a compression appears in the elastic response.*

3.3. **Discussion.**

3.3.1. *Dependence of the curves on the characteristic lengths d_c and ℓ.* We assume here that the critical stress σ_c is fixed and study the dependence of the Dugdale's branches on d_c at fixed ℓ, or, on ℓ at fixed d_c. Therefore, in any case, the loading t_e at which a cohesive crack nucleates is fixed and hence the elastic branch is always the same.

• *At fixed ℓ.* For all d_c, the fully cohesive branch is a part of the parabola $a = \ell\sqrt{1 - t_e/t}$. Only the final point (t_i, a_i) depends on d_c, and both t_i and a_i are increasing functions of d_c (or ϵ), see (34)-(35) and

Figure 6. On the one hand, when d_c (or ϵ) goes to 0, then t_i tends to t_e and a_i/ℓ tends to 0 like $\epsilon^{1/3}$. On the other hand, when d_c/ℓ goes to infinity, then a_i tends to ℓ and t_i tends to infinity. That means that the smaller the material length d_c, the weaker the stabilizing effect of the stress gradient.

In the same manner, for the partially non cohesive branch, the smaller the material length d_c, the more accentuated the snap-back and the larger the size of the loop. When d_c tends to 0, the parameter α_m of the lowest point of the loop tends to $1/e$ and hence the time t_m tends to $\frac{2t_e}{\pi}\arccos(1/e) \approx 0.760\, t_e$ whereas $a_m \approx e d_c$ tends to 0 like d_c. The crack length a_l of the limit point tends to $\ell/\sqrt{5}$ whereas the time $t_l \approx \frac{5^{5/4}}{\pi}\sqrt{\frac{d_c}{2\ell}}\, t_e$ tends to 0 like $\sqrt{d_c}$.

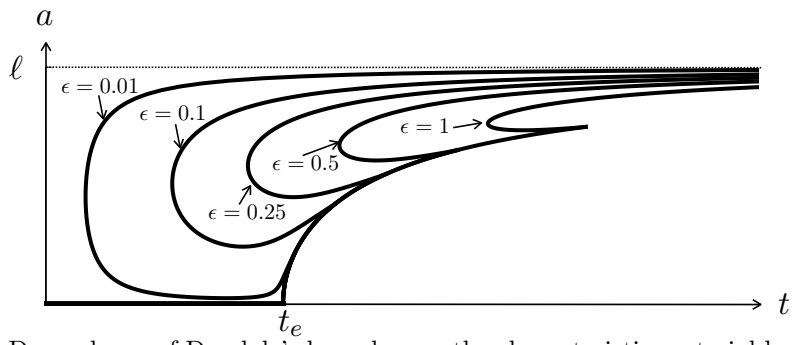


FIGURE 6. Dependence of Dugdale's branches on the characteristic material length $d_c = \epsilon\ell$ at fixed stress gradient length ℓ.

• *At fixed d_c.* For a given material, one can see the influence of the intensity of the stress gradient by comparing on Figure 7 the Dugdale branches associated with different values of ℓ. Let us recall that the higher the stress gradient, the smaller the length ℓ, the case of a uniform stress field corresponding to $\ell = +\infty$. Accordingly, the higher the gradient, the greater the fully cohesive branch, the smaller the loop of the partially non cohesive branch and the smaller the final length of the crack. For small stress gradient and hence large ℓ and small ϵ, the asymptotic behaviors when ϵ goes to 0 are the same as those presented above for fixed ℓ.

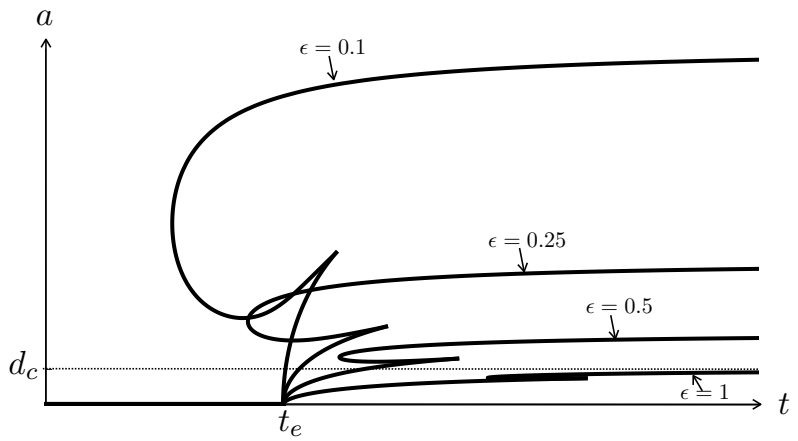


FIGURE 7. Dependence of Dugdale's branches on the stress gradient length $\ell = d_c/\epsilon$ at fixed characteristic material length d_c.

3.3.2. *The response under monotonically increasing loading.* If the body is submitted to an increasing proportional loading starting from $t = 0$ and growing to infinity, then the response is purely elastic as long as $t \leq t_e$. In the interval (t_e, t_i), a cohesive crack nucleates and its length continuously grows since the fully cohesive branch $a(t)$ is monotonically increasing. At $t = t_i$, the length of the cohesive crack is a_i and its maximal opening (located at $x_1 = 0$) reaches the critical value δ_c at which the cohesive forces disappear. As soon as the loading becomes greater than t_i, necessarily a non cohesive zone zone must appear at the center of the crack. But since the partially non cohesive branch suffers a snap back, the response cannot follow its loop and the length of the crack must be discontinuous at t_i. If one neglects inertial effects, then the unique possible stable configuration corresponds to a partially non cohesive crack of total length a_i^* located on the upper part of the partially non cohesive branch, cf Figure 8.

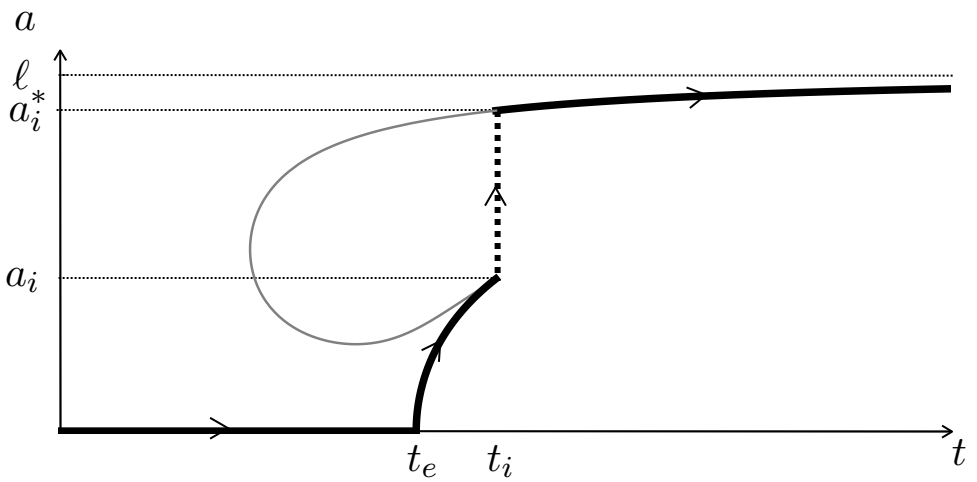


FIGURE 8. Evolution of the crack length under a monotonic increasing loading when $d_c/\ell = 0.1$. Note the jump at $t = t_i$.

Of course, the fact that one can neglect the inertial effects while the crack length suffers a jump discontinuity should be justified by a careful dynamical analysis. An alternative option would be to consider that the jump of the crack length is governed by a principle of energy conservation. Such a study is outside the scope of the present work and the interested reader should refer to [Dumouchel et al., 2007; Charlotte et al., 2008; Lazzaroni et al., 2012] for a complete analysis of dynamical crack propagation in the framework of Griffith's theory. Accordingly, if one adopts the assumption that the inertial effects are negligible, then the amplitude of the jump depends in particular on $\epsilon = d_c/\ell$: the smaller ϵ, the greater the jump. Specifically, at given material length d_c, the smaller the gradient stress length d_c, the greater the jump. On the one hand, for large values of ℓ (small stress gradient) and hence small values of ϵ, a_i/ℓ is small and of the order of $\epsilon^{1/3}$ whereas a_i^* is large and practically equal to ℓ. That means that the nucleation of a crack is brutal and the stabilizing effect of gradient stress is weak. On the other hand, for values of ℓ of the same order as d_c, the jump is weak and the stabilizing effect of gradient stress is stronger, see Figure 7.

Remark 10. *It would seem that the shape of the loop and the snap-back part of the partially non cohesive branch do not play any role in the crack propagation under monotonic loading. Moreover one could believe that the snap-back part is a purely mathematical byproduct of our modeling where the irreversibility of the crack propagation is not taken into account. Indeed, even if one decreases the loading just after the end of the fully non cohesive branch, it is not physically admissible to follow the partially non cohesive branch because both crack lengths a and b should decrease. In fact, we will show in the next paragraph that the loop can be observed and even that it plays a fundamental role in presence of imperfections.*

3.3.3. *Sensibility to the imperfections.* Up to now, all the analysis is made in the ideal case where the body is homogeneous and does not contain any defect before the loading process. Such a situation will be called the perfect case by opposition to the case of preexisting defects. In the present paper we will only consider the case where the imperfection corresponds to an initial cut along the $x_2 = 0$ axis, centered at 0 and of half-length $a_0 < \ell$. In other words, we assume that the body contains a preexisting *non cohesive* crack $(-a_0, a_0) \times \{0\}$ whose length is a parameter. Accordingly, the elastic response is no more regular, but the stress is singular at the tips $\pm a_0$ as soon as a loading is applied. Therefore, by virtue of Dugdale's model and Proposition 2, there exists no more an elastic branch, but a cohesive zone must nucleate ahead the tips $\pm a_0$ as soon as $t > 0$ with a length $a - a_0$ such that the singularity vanishes at the tips $\pm a$. Assuming that the initial crack is small by comparison to the size of the body, *i.e.* $a_0 \ll L$, one can still follow the two-scale approach presented in Section 3.2. In particular we can use the expressions (39), at time t, for the stress intensity factor at the tips $\pm a$ of a crack whose non cohesive zones are of length b, namely $\mathsf{K_I}[t, a, b]$. Therefore, the relation between a, b and t in order that the singularity vanishes remains given by (40). Similarly, the opening at the tips $\pm b$, namely $[\![u[t, a, b]_2]\!](b)$, is still given by (42). Equipped with those two relations, it is easy to determine the evolution of the preexisting crack under a monotonically increasing loading. Specifically, the evolution can be divided into the two or three following parts, according to the value of a_0:

(1) **Cohesive phase:** *Growth of two symmetric purely cohesive zones, the non cohesive crack tips remaining at* $\pm a_0$. For t small enough, the initial non cohesive crack does not propagate because the opening at $\pm a_0$ remains less than δ_c, but two symmetric cohesive zones grow in order to cancel the singularity at those points. The relation between a and t is given by the condition $\mathsf{K_I}[t, a, a_0] = 0$ and hence (40) with $b = a_0$ leads to

$$\frac{t}{t_e} = \frac{2}{\pi} \frac{\arccos \dfrac{a_0}{a}}{\left(1 - \dfrac{a^2}{\ell^2}\right)}. \tag{47}$$

Since (47) gives a monotonically increasing relation between t and a when $a \in [a_0, \ell)$, the relation is invertible and hence a is an increasing function of t starting from a_0 at $t = 0$. That allows us to define the so-called cohesive branch associated with the initial crack length a_0 in the diagram (t, a). Moreover, for a and t satisfying (47), (42) with $b = a_0$ gives

$$[\![u[t, a, a_0]_2]\!](a_0) = 8(1 - \nu^2) \frac{\sigma_c}{E} \left(\frac{1}{3} \frac{t}{t_e} \frac{(a^2 - a_0^2)^{3/2}}{\ell^2} + \frac{a_0}{\pi} \ln \frac{a}{a_0} \right),$$

and hence the opening at $\pm a_0$ is an increasing function of t starting from 0 at $t = 0$. By construction, it will reach the critical value δ_c when the triple (a, a_0, t) satisfies both (40) and (43). Therefore that triple is the point of the partially non cohesive branch of the perfect case which corresponds to $b = a_0$. The associated parameter α_0 is given by the equation

$$\bar{b}^\epsilon(\alpha_0)\ell = a_0,$$

its uniqueness being ensured by the monotonicity of the function $\bar{b}^\epsilon(\alpha)$. In other words the cohesive branch will finish when it intersects the loop of the perfect case. In conclusion, the cohesive branch starts from $(0, a_0)$ and finishes at $(\bar{t}^\epsilon(\alpha_0)t_e, \bar{a}^\epsilon(\alpha_0)\ell)$. During this phase, the total crack length and the opening of any point of the crack lips increase with t and hence there exists no incompatibility with an irreversibility condition.

(2) **Possible jump of the crack length:** *Brutal propagation of the crack if the cohesive branch intersects the lower part of the loop of the perfect case.* If the final point of the cohesive branch is lower than the limit point of the loop, *i.e.* if $\bar{a}^\epsilon(\alpha_0)\ell < a_l$, then the crack evolution will suffer a jump. Indeed, the evolution cannot follow the lower part of the loop in the direction of increasing time

for obvious irreversibility reasons since the crack length should decrease. Therefore, if one neglects inertial effects, the unique possibility is that the evolution be discontinuous and the point just after the jump be the point located at the same time $\bar{t}^\epsilon(\alpha_0)t_e$ on the upper part of the loop. On the other hand, if the final point of the cohesive branch is at or above the limit point of the loop, *i.e.* if $\bar{a}^\epsilon(\alpha_0)\ell \geq a_l$, then the evolution can continuously follow that part of the curve in the sense of increasing time since the crack length increases and no jump is necessary.

(3) **The continuous growth of a partially non cohesive crack.** Once the upper part of the loop has been reached, which can be arrived after a jump, the evolution of the crack simply follow that upper part of the loop in the direction of increasing time and finally the crack length will tends to ℓ when t goes to infinity as in the perfect case. The system will finally forget its initial imperfection.

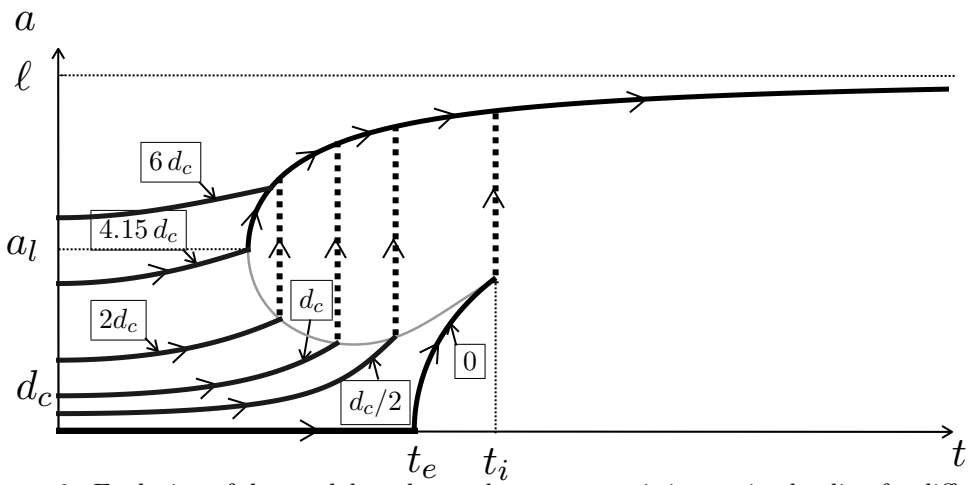


FIGURE 9. Evolution of the crack length a under a monotonic increasing loading for different lengths a_0 of the centered initial non cohesive crack. Here, $d_c/\ell = 0.1$ and $a_0 = 0, d_c/2, d_c, 2d_c, 4.15\,d_c, 6d_c$.

All these results can be seen on Figure 9 where are considered five cases of imperfection size. The first three, which correspond to a small initial crack length, lead to a jump whereas the last two, corresponding to a sufficiently large initial crack length, give rise to a continuous growth of the crack. Of course, the critical length of the initial crack above which the evolution is continuous depends both on ℓ and d_c. In any case, one sees the fundamental role played by the loop of the perfect system.

3.3.4. *Comparison with Griffith's theory.* To finish this discussion let us compare the evolution predicted by Dugdale's model with the one associated with Griffith's theory. To this purpose, let us consider an initial centered (non cohesive) crack of half-length $a_0 \ll L$ and let us determine for which loading t_0 that crack will propagate if one uses Griffith's criterion. Since the initial crack is small, one can use the results of the two-scale approach. Since there is no cohesive forces in Griffith's theory, the stress field is singular at the tips $\pm a_0$ of the crack and the stress intensity factor is given by (39) with $b = a_0$. Accordingly, one gets

$$\mathsf{K_I}[t_0, a_0, a_0] = \sigma_c\sqrt{\pi a_0}\left(1 - \frac{a_0^2}{\ell^2}\right)\frac{t_0}{t_e}$$

and hence, by virtue of Irwin's formula, the energy release rate G in a plane strain setting reads as

$$G = \pi(1-\nu^2)\frac{\sigma_c^2}{E}a_0\left(1 - \frac{a_0^2}{\ell^2}\right)^2\frac{t_0^2}{t_e^2}.$$

The Griffith criterion $G = G_c$ gives the loading t_0 at which the initial crack of half-length a_0 will propagate. Specifically after identifying G_c with that of Dugdale's model, t_0 and a_0 are related by

$$\frac{t_0}{t_e} = \frac{2\sqrt{2}}{\pi}\sqrt{\frac{d_c}{a_0}}\frac{\ell^2}{\ell^2 - a_0^2}.$$

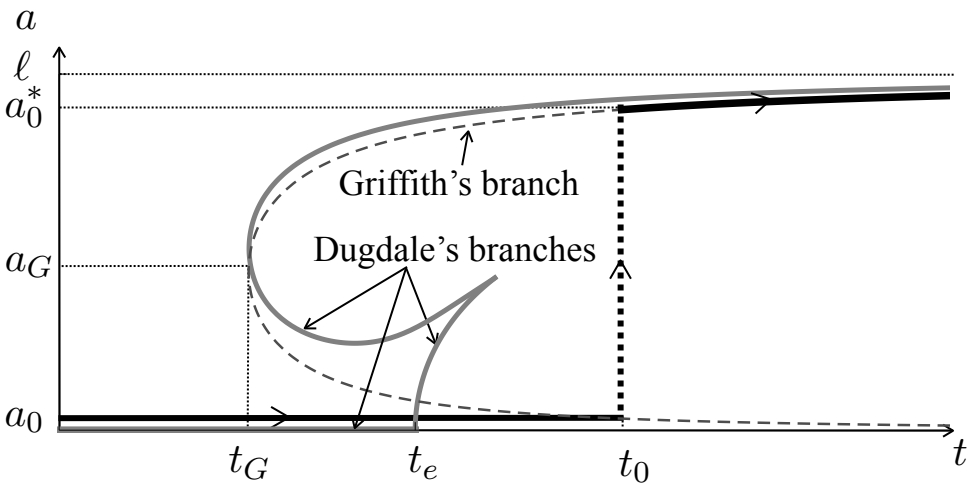


FIGURE 10. Comparison of Dugdale's branches with Griffith's branch when $d_c/\ell = 0.1$. In dark, the response associated with Griffith's law under monotonic loading when the body contains an initial crack of half-length a_0.

The graph of this relation in the diagram (t, a) is divided into two branches, both starting from the point (t_G, a_G) with

$$t_G = \frac{5^{5/4}}{\pi\sqrt{2}}\sqrt{\epsilon}\, t_e, \qquad a_G = \frac{\ell}{\sqrt{5}}.$$

Along the upper branch, above (t_G, a_G), t_0 increases to infinity when a_0 increases to ℓ, whereas along the lower branch, below (t_G, a_G), t_0 increases to infinity when a_0 decreases to 0. Accordingly, when $a_0 < a_G$, the smaller the initial crack, the greater the loading at which it propagates. At the limit no crack can nucleate in a sound body, what is one of the main drawbacks of Griffith's theory. When the initial crack is such that $0 < a_0 < a_G$ and if one neglects inertial effects, the crack length will jump instantaneously at t_0 to the associated point a_0^* on the upper Griffith's branch. Then, the evolution will propagate continuously by following the upper Griffith's branch, see Figure 10.

Let us compare with Dugdale's law. For small values of ϵ, the upper part of the partially non cohesive branch of Dugdale's model is close to the upper part of Griffith's branch. In particular, when ϵ tends to 0, a_l tends to a_G, the ratio t_l/t_G tends to 1 whereas both t_l and t_G go to 0 like $\sqrt{\epsilon}$. But the lower part of the partially non cohesive branch and the fully cohesive branch of Dugdale's model remains different from the lower part of Griffith's branch. In particular, the loading at which a crack nucleates or a preexisting crack propagates cannot be greater than t_e with Dugdale's model whereas it is not bounded but strongly dependent on the size of the preexisting crack with Griffith's model. That means that the nucleation and the first phase of the propagation of a crack are strongly different according to one uses Griffith or Dugdale model. But once the crack length is large by comparison to the Dugdale characteristic length d_c, then the cohesive zones become negligible and the two models give practically the same results.

4. Conclusion and Perspectives

Let us summarize the main results obtained in this paper. First, since Dugdale's law contains a critical stress σ_c, one can account for the nucleation of a crack in a sound body at a finite loading t_e, in contrast with Griffith's law. However, only the first phase of the nucleation, that one where the whole crack is submitted to cohesive forces, leads to a continuous evolution of the crack length with the loading. Indeed, at the loading t_i when the opening reaches the critical value δ_c, the evolution is necessarily discontinuous and leads to a jump of the crack length because of the presence of a snap-back in the equilibrium branch. Moreover, since Dugdale's model contains also a material characteristic length ℓ, size effects are possible. Assuming that ℓ is small by comparison to the size of the body, situation the most frequent in practice, all the solutions can be obtained in a closed form which renders easy the study of the size effects. In particular, one shows that, in presence of stress gradient, the response is very sensitive to the ratio ϵ between the material length d_c and the characteristic length ℓ of the stress gradient. The higher the stress gradient, the smaller the length ℓ, the shorter the first phase of nucleation and the greater the jump of the crack length at t_i which is of the order of ℓ. Accordingly, the loading t_i at which the jump occurs can be considered as the loading at which a "macroscopic" crack nucleates in the body. The formula (35) which gives t_i can be considered as universal, in the sense of not dependent on the particular problem (but specific to Dugdale's model), provided that ϵ is small. Finally, the snap-back in the partially non cohesive branch, which is also obtained in a closed form and whose first part can be also considered as universal, plays an important role in presence of imperfections.

All these results which have been obtained for Dugdale's model should be extended for more general cohesive models. The two-scale approach can be followed in any case, but the difficulty will be to solve the different problems in a closed form. The help of numerical methods could be necessary.

Appendix A. The generic local problem and its solving

Let us consider the following plane elastic problem which is set on the entire plane except a crack of length a centered at the origin in the x_1 direction:

$$\begin{cases} \operatorname{div}\boldsymbol{\sigma} = \mathbf{0} & \text{in} \quad \mathbb{R}^2 \setminus ([-a,a]\times\{0\}) \\ \boldsymbol{\sigma} = \lambda \operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) & \text{in} \quad \mathbb{R}^2 \setminus ([-a,a]\times\{0\}) \\ \boldsymbol{\sigma}(\mathbf{x}) \to \mathbf{0} & \text{when} \quad |\mathbf{x}| \to \infty, \\ \boldsymbol{\sigma}(x_1,0)\mathbf{e}_2 = \mathtt{T}(x_1)\mathbf{e}_2 & \text{when} \quad x_1 \in (-a,a) \end{cases} \tag{48}$$

where $\mathtt{T}(x_1)$ represents the normal force distribution on the lips of the crack. The solution which is defined up to an arbitrary rigid displacement can be found by using complex potentials, cf [Muskhelishvili, 1963]. Specifically, the components of the displacement and of the stress are given in terms of the complex potential $\varphi(z)$, $z = x_1 + \mathtt{i}x_2$. Specifically, one has

$$\sigma_{22}(x_1,x_2) - \mathtt{i}\sigma_{12}(x_1,x_2) = \varphi'(z) + \varphi'(\overline{z}) + (z-\overline{z})\overline{\varphi''(z)}, \tag{49}$$

$$2\mu\big(u_1(x_1,x_2) + \mathtt{i}u_2(x_1,x_2)\big) = (3-4\nu)\varphi(z) - \varphi(\overline{z}) - (z-\overline{z})\overline{\varphi'(z)}, \tag{50}$$

φ being holomorphic in the plane without the crack, the bar denoting the complex conjugate. By a standard procedure, we get

$$\varphi'(z) = \frac{1}{2\pi\sqrt{z^2-a^2}} \int_{-a}^{a} \frac{\mathtt{T}(s)\sqrt{a^2-s^2}}{s-z} ds, \tag{51}$$

from which one deduces the normal stress distribution along the axis $x_2 = 0$. Specifically, outside the lips of the crack, one gets

$$\sigma_{22}(x_1,0) = \frac{1}{\pi\sqrt{x_1^2-a^2}} \int_{-a}^{a} \frac{\mathtt{T}(s)\sqrt{a^2-s^2}}{s-x_1} ds \quad \text{when} \quad |x_1| > a. \tag{52}$$

Therefore the normal stress is in general singular at the tips of the crack with a singularity of the type $\sigma_{22}(a+r,0) \sim \mathsf{K_I}/\sqrt{2\pi r}$ for r close to 0. Accordingly, the relation between the mode I stress intensity factor $\mathsf{K_I}$ at $\pm a$ (which is the same at the two tips by symmetry) and the normal stress repartition $\mathtt{T}$ is given by

$$\mathsf{K_I} = -\frac{1}{\sqrt{\pi a}} \int_{-a}^{a} \mathtt{T}(s) \sqrt{\frac{a+s}{a-s}} ds. \tag{53}$$

After an integration of (51), one obtains the jump of the normal displacement across the crack (the arbitrary rigid displacement does not play any role):

$$[\![u_2]\!](x_1) = \frac{4(1-\nu^2)}{\mathtt{i}E} [\![\varphi]\!](x_1). \tag{54}$$

References

[Abdelmoula et al., 2009a] Abdelmoula, R., J.-J. Marigo, and T. Weller: 2009a, 'Construction of a fatigue law from a cohesive force model: the mode III case'. *Comptes Rendus Mécanique* **337**(1), 53–59.

[Abdelmoula et al., 2009b] Abdelmoula, R., J.-J. Marigo, and T. Weller: 2009b, 'Construction of fatigue laws from cohesive forces models: the mode I case.'. *Comptes Rendus Mécanique* **337**(3), 166–172.

[Abdelmoula et al., 2010] Abdelmoula, R., J.-J. Marigo, and T. Weller: 2010, 'Construction and justification of Paris-like fatigue laws from Dugdale-type cohesive models'. *Annals of Solid and Structural Mechanics* **1**(3-4), 139–158.

[Barenblatt, 1962] Barenblatt, G.: 1962, 'The methematical theory of equilibrium cracks in brittle fracture'. *Adv. Appl. Mech.* **7**, 55–129.

[Bourdin et al., 2008] Bourdin, B., G. A. Francfort, and J.-J. Marigo: 2008, 'The variational approach to fracture'. *J. Elasticity* **91**(1–3), 5–148.

[Bui, 1978] Bui, H. D.: 1978, *Mécanique de la rupture fragile*. Masson.

[Charlotte et al., 2008] Charlotte, M., P.-E. Dumouchel, and J.-J. Marigo: 2008, 'Dynamic fracture: an example of convergence towards a discontinuous quasi-static solution'. **20**, 1–19.

[Charlotte et al., 2000] Charlotte, M., G. A. Francfort, J.-J. Marigo, and L. Truskinovsky: 2000, 'Revisiting brittle fracture as an energy minimization problem : comparison of Griffith and Barenblatt surface energy models'. In: A. Benallal (ed.): *Continuous Damage and Fracture*, The data science library. Elsevier, pp. 7–18.

[Charlotte et al., 2006] Charlotte, M., J. Laverne, and J.-J. Marigo: 2006, 'Initiation of cracks with cohesive force models: a variational approach'. *Eur. J. Mech. A/Solids* **25**(4), 649–669.

[Dang et al., 2014] Dang, T. B. T., L. Halpern, and J.-J. Marigo: 2014, 'Asymptotic analysis of small defects near a singular point in anti-plane elasticity. Application to the nucleation of a crack at a notch'. *Mathematics and Mechanics of Complex Systems* **2**(2), 141–179.

[Dang et al., 2013] Dang, T. B. T., J.-J. Marigo, and L. Halpern: 2013, 'Matching Asymptotic Method in Propagation of Cracks with Dugdale Model'. *Key Engineering Materials* **525-526**, 489–92.

[Del Piero, 1999] Del Piero, G.: 1999, 'One-Dimensional ductile-brittle transition, yielding and structured deformations'. *in: P. Argoul, M. Frémond (Eds.), Proceedings of IUTAM Symposium "Variations de domaines et frontières libres en mécanique", Paris, 1997, Kluwer Academic.*

[Del Piero and Raous, 2010] Del Piero, G. and M. Raous: 2010, 'A unified model for adhesive interfaces with damage, viscosity, and friction'. *Eur. J. Mech. A/Solids* **29**(4), 496–507.

[Dugdale, 1960] Dugdale, D.: 1960, 'Yielding of steel sheets containing slits'. *J. Mech. Phys. Solids* **8**, 100–108.

[Dumouchel et al., 2007] Dumouchel, P.-E., J.-J. Marigo, and M. Charlotte: 2007, 'Rupture dynamique et fissuration quasi-statique instable'. *Comptes Rendus Mécanique* **335**(11), 708–713.

[Ferdjani et al., 2007] Ferdjani, H., R. Abdelmoula, and J.-J. Marigo: 2007, 'Insensitivity to small defects of the rupture of materials governed by the Dugdale model'. *Continuum Mech. Thermodyn.* **19**(3-4), 191–210.

[Ferdjani et al., 2009] Ferdjani, H., R. Abdelmoula, J. J. Marigo, and S. El Borgi: 2009, 'Study of size effects in the Dugdale model through the case of a crack in a semi-infinite plane under anti-plane shear loading'. *Continuum Mech. Thermodyn.* **21**(1), 41–55.

[Giacomini, 2005] Giacomini, A.: 2005, 'Size effects on quasi-static growth of cracks'. *SIAM J. Math. Anal.* **36**(6), 1887–1928.

[Griffith, 1920] Griffith, A.: 1920, 'The phenomena of rupture and flow in solids'. *Philos. Trans. Roy. Soc. London* **CCXXI(A)**, 163–198.

[Grisvard, 1985] Grisvard, P.: 1985, *Elliptic problems in non smooth domains*, No. 24 in Monographs and Studies in Mathematics. Pitman.

[Jaubert and Marigo, 2006] Jaubert, A. and J.-J. Marigo: 2006, 'Justification of Paris-type Fatigue Laws from Cohesive Forces Model via a Variational Approach'. *Continuum Mech. Thermodyn.* **V18**(1), 23–45.

[Keller et al., 1999] Keller, K., S. Weihe, T. Siegmund, and B. Kroplin: 1999, 'Generalized Cohesive Zone Model: incorporating triaxiality dependent failure mechanisms'. *Computational Materials Science* pp. 267–274.

[Laverne and Marigo, 2004] Laverne, J. and J.-J. Marigo: 2004, 'Approche globale, minima relatifs et Critère d'Amorçage en Mécanique de la Rupture'. *Comptes Rendus Mecanique* **332**(4), 313–318.

[Lazzaroni et al., 2012] Lazzaroni, G., R. Bargellini, P.-E. Dumouchel, and J.-J. Marigo: 2012, 'On the role of kinetic energy during unstable propagation in a heterogeneous peeling test'. *International Journal of Fracture* **175**, 127–150. Isiweb.

[Lorentz, 2008] Lorentz, E.: 2008, 'A mixed interface finite element for cohesive models'. *Comput. Methods Appl. Mech. Engrg.* **198**, 302–317.

[Marigo and Truskinovsky, 2004] Marigo, J. and L. Truskinovsky: 2004, 'Initiation and propagation of fracture in the models of Griffith and Barenblatt'. *Continuum Mech. Thermodyn.* **16**(4), 391–409.

[Muskhelishvili, 1963] Muskhelishvili, N. I.: 1963, *Some basic problems of mathematical theory of elasticity*. P. Noordhoff Ltd, Groningen.

[Needleman, 1992] Needleman, A.: 1992, 'Micromechanical modelling of interface decohesion'. *Ultramicroscopy* **40**, 203–214.

[Nguyen et al., 2001] Nguyen, O., E. A. Repetto, M. Ortiz, and R. A. Radovitzki: 2001, 'A cohesive model of fatigue crack growth'. *Eng. Fract. Mech.* **110**, 351–369.

[Paris et al., 1961] Paris, P. C., M. P. Gomez, and W. E. Anderson: 1961, 'A rational analytic theory of fatigue'. *The Trend in Engineering* **13**(8), 9–14.

[Roe and Siegmund, 2002] Roe, K. L. and T. Siegmund: 2002, 'An irreversible cohesive zone model for interface fatigue crack growth simulation'. *Eng. Fract. Mech.* **70**, 209–232.

[Talon and Curnier, 2003] Talon, C. and A. Curnier: 2003, 'A model of adhesion coupled to contact and friction'. *Eur. J. Mech. A/Solids* **22 (4)**, 545–565.

[Tvergaard, 1990] Tvergaard, V.: 1990, 'Effect of fiber debonding in a whisker-reinforced metal'. *Mat. Sci. Eng. A-Struc.* **125**, 203–213.

[Willis, 1967] Willis, J. R.: 1967, 'A comparison of the fracture criteria of Griffith and Barenblatt'. *J. Mech. Phys. Solids* **15**, 151–162.

(T. H. Pham) CNRS, Ecole Polytechnique, UMR 7649, Laboratoire de. Mécanique des Solides, F-91128 Palaiseau Cedex, France

E-mail address, T. H. Pham: `tuan-hiep.pham@polytechnique.org`

(J. Laverne) EDF R& D, France

E-mail address, J. Laverne: `jerome.laverne@edf.fr`

(J.-J. Marigo) CNRS, Ecole Polytechnique, UMR 7649, Laboratoire de. Mécanique des Solides, F-91128 Palaiseau Cedex, France

E-mail address, J.-J. Marigo: `marigo@lms.polytechnique.fr`