Optimal insurance coverage of low probability-high severity risks

Alexis Louaas, Pierre Picard

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Abstract: Catastrophic risks are often characterized by a low probability and a high severity. Taking these specificities into account, we analyze the intrinsic reasons for which catastrophic risks may be more or less insurable, independently from the market failures frequently observed in practice. On the demand side, we characterize individual preferences under which the willingness to pay for the coverage of large losses remains significant, although their occurrence probability is very small. On the supply side, the correlation between individual losses affects the insurance pricing through the insurers' cost of capital. Analyzing the interaction between demand and supply yields the key determinants of insurability and of a socially optimal risk sharing strategy.

Keywords: Disaster insurance, catastrophic risk, risk aversion, capital costs.

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1 Introduction

The insurability of catastrophic risks is at the heart of many economic policy debates and the aggregate consequences of these risks, be they natural or man-made, are often emphasized. Rietz (1988) and Barro (2006) for example, documented significant effects on asset prices while Gourio (2012) and Farhi & Gabaix (2015) highlighted effects on business cycles and interest rates. In the context of global climate change, Weitzman (2009) also illustrated the potentially disastrous consequences of uninsured catastrophic risks on welfare. The coverage of these risks is therefore a crucial question, at the intersection of government decision making and market mechanisms.

In practice, catastrophic risks often remain poorly insured, as shown by an important body of literature following Kunreuther (1973). The explanations for these low coverage rates have ranged from departures from the expected utility paradigm (Kunreuther & Slovic (1978), Kunreuther et al. (2001) and Hertwig et al. (2004)) to the crowding-out of private insurance demand by public relief (Raschky et al. (2013), Kousky et al. (2013) and Grislain-Letrémy (2018)). Imperfect capital markets are also often presented as a significant impediment to the insurability of catastrophic risks (Jaffee & Russell (1997), Froot (2001), Niehaus (2002), Zanjani (2002), Cummins et al. (2002) and Kousky & Cooke (2012)).

If market failures are needed to justify government interventions, they are not necessary *a priori* to rationalize low insurance take-up rates. As we will show, covering risks that have a systemic component is, by nature, particularly costly, even in a perfectly competitive set-up. Disentangling the role of market failures from the intrinsic characteristics of the risk that may make it uninsurable is therefore an important pre-requisiste to adequate policy intervention.

The present paper consequently remains upstream of market failures and departures from expected utility, and our objective is to highlight the intrinsic determinants of demand and supply in the insurance market for catastrophic risks. To some extent, this does not differ from a standard market analysis, where explanatory factors determine demand and supply. However, as we will see, the insurance coverage of events that are, at the same time, very rare (low probability) and simultaneously highly damaging to many people (high severity) requires a specific approach. On the demand side, the severity of the losses incurred by individuals makes insurance particularly valuable, but low probabilities reduce the propensity of individuals to purchase insurance. The willingness to pay for the coverage of such risks results from these counter-balancing effects. On the supply side, the fact that many individuals are affected at the same time when a disaster occurs makes insurance costlier to provide since capital must be raised to maintain the insurer's solvency. Our objective in this paper is to analyze these specificities, in order to bring out the key determinants of insurance coverage for catastrophic risks. We find that these key determinants are the degree of risk aversion that individuals feel when they experience the highest possible loss and the correlation between individual risks.

Our analysis shows that, in a perfectly competitive insurance market without transaction costs, low probability risks that entail serious losses for victims are (at least partially) covered, in spite of their systemic nature. From a positive economics perspective, this suggests that substantial market failures or transaction costs are likely to play a key role in the explanation of low catastrophic insurance take-up rates. From a normative standpoint, our analysis emphasizes the favorable welfare effects of policies that seek to organize insurance schemes against catastrophic risks by lowering transaction costs or by mitigating the various form of market failures that may be at stake.

As a preliminary step, we investigate how low probability - high severity risks can be viewed through the lens of insurance microeconomics. To do so, we first extend the Arrow (1963) and Pratt (1964) approximation of the risk premium to account for potentially large deviations from the mean and we provide an asymptotic characterization of the willingness to pay for a very low probability risk. We show that, when individuals display decreasing absolute risk aversion (DARA), a high absolute risk aversion (or, equivalently, a low risk tolerance) in the accident state may entail a significant willingness to pay to avoid risk, even if the accident probability is very low (an extreme case being infinite absolute risk aversion in the loss state, as in Weitzman (2009)).

We then investigate the optimal insurance coverage of an individual who faces the risk of an accident with a very low probability. We extend the canonical model of the optimal insurance literature (Mossin (1968) and Raviv (1979)) by considering a general insurance pricing rule, and we characterize the asymptotic optimal insurance coverage when the loss probability tends to zero.

We subsequently study the supply side of the insurance market by considering the risk of a low probability catastrophe affecting a large number of individuals at the same time. To maintain its solvency, the insurance provider raises capital on financial markets and, since the catastrophe is systemic, capital is costly to obtain and features a positive risk premium. This cost of capital is passed onto the policyholders through the insurance premiums, that are therefore above the actuarially fair prices. Low levels of coverage rates may consequently occur in equilibrium when the catastrophe is highly systemic, even in a complete financial market setting and in the absence of other forms of market failures. We show however, that except for the case of perfectly correlated losses, the optimal insurance coverage remains positive for low probability risks.

Our conclusions may also be presented in a more normative perspective, by focusing on their policy implications. Removing the barriers to catastrophic risk insurability is a multi-faceted challenge, including the promotion of adequate financial innovations, and the targeting of government policy towards risk prevention and the assistance of the most vulnerable groups. Our approach is restricted to the preliminary question as to whether, and under which conditions, the coverage of catastrophic risks by insurance mechanisms is a socially optimal objective. As shown in this paper, the answer to this question is positive if the perspective of incurring a large loss leads the individuals to display a degree of risk aversion that is large enough to compensate the high price of insurance induced by the systemic nature of the risk. This condition would always be satisfied if there were no transaction costs associated with the insurance mechanisms itself (e.g., distribution and claim handling costs). Reducing transaction costs is thus a crucial step toward insurability. Put differently, in the presence of large transaction costs, assuming as a matter of principle, that individuals should be protected by state-sponsored insurance mechanisms, whatever their willingness to pay and the cost of capital, is not the appropriate way to approach the catastrophic risk insurance issue.

Finally, we complete these theoretical foundations by simulating the optimal insurance coverage. Our simulations yield significant levels of coverage. We also show that the optimal coverage converges rapidly to its limit, which corroborates the relevance of the asymptotic approach used in the previous sections. This also suggests that differences in accident probabilities between individuals may result in relatively small differences in the optimal coverage choices (as long as all loss probabilities are sufficiently low). This result has a practical importance to design collective insurance schemes against low probability - high severity disasters when people face different loss probabilities due, for example, to the geographical distance between their residential areas and the source of a potential risk (e.g., a nuclear power plant).

The rest of the paper is organized as follows. Section 2 and 3 respectively analyze the risk premium and the insurance demand for a low probability - high severity risk and Section 4 illustrates the relevance of our results with numerical simulations. Section 5 concludes by summarizing our results and relating them to public policy issues. Section 6 gathers the proofs.

2 The risk premium of low-probability and high-severity risks

The Arrow (1963) and Pratt (1964) approximation of the risk premium holds for small risks, with little variation around the mean. As a preliminary analysis of our study of optimal insurance against catastrophic risks, this section extends the characterization of the risk premium to high severity risks.

Consider an expected utility risk-averse individual with a von Neumann-Morgenstern utility function $u(x)$ such that $u' > 0$ and $u'' < 0$, where x is the individual's wealth. Let $A(x) = -u''(x)/u'(x)$ and $T(x) = 1/A(x)$ be his indices of absolute risk aversion and of risk tolerance, respectively. He holds an initial wealth w, and faces the risk of a loss $L < w$ with probability p. Thus $m(p, L) = pL$ and $\sigma^2(p, L) = p(1-p)L^2$ are the expected loss and the variance of the loss, respectively. The certainty equivalent $C(p, L)$ of this lottery is defined by

$$u(w - C) = (1-p)u(w) + pu(w - L).$$

Straightforward calculations give

$$\begin{aligned} C'_p(p, L) &= \frac{u(w) - u(w-L)}{u'(w-C)} > 0, \\ C''_{p^2}(p, L) &= -C'_p(p, L)^2 A(w - C) < 0. \end{aligned}$$

Thus, $C(p, L)$ is increasing and concave with respect to p, and of course we have $C(0, L) = 0$. We also denote

$$\theta(p, L) \equiv \frac{C(p, L) - m(p, L)}{\sigma^2(p, L)},$$

the normalized risk premium, that is the risk premium per unit of variance.

Put informally, the risk (p, L) may be considered catastrophic for the individual if $C(p, L)$ is non-negligible although p is very small. Risk aversion implies that $C(p, L) > pL$. L'Hôpital's rule allows us to write the limit ratio of certainty equivalent to expected loss as

$$\lim_{p \to 0} \frac{C(p, L)}{pL} = \frac{C'_p(0, L)}{L},$$

which is proportional to $C'_p(0, L)$ for L given. Using l'Hôpital's rule again gives

$$\theta(0, L) \equiv \lim_{p \longrightarrow 0} \theta(p, L) = \frac{C'_p(0, L) - L}{L^2}. \tag{1}$$

Thus, analyzing the determinants of $\theta(0, L)$ is an intermediate step to understanding why $C_p'(0, L)$ may be large and thus why $C(p, L)$ may be significant although p is very small.

When L is small, we know from the Arrow-Pratt approximation that the risk premium per unit of variance is proportional to the index of absolute risk aversion. Indeed, we have

$$\lim_{L \longrightarrow 0} \theta(p, L) = \frac{A(w)}{2} \text{ for all } p \in (0, 1),$$

which of course also holds when p goes to 0, that is

$$\lim_{L \longrightarrow 0} \theta(0, L) = \frac{A(w)}{2}.$$

When L is large, it is intuitive that the size of the risk premium depends on function $A(x)$ not only in the neighborhood of $x = w$, but over the whole interval $[w - L, L]$. This is confirmed by Proposition 1 and its corollaries. Proposition 1 provides an exact formula for $\theta(0, L)$ which is a weighted average of $A(x) \exp\{\int_x^w A(t)dt\}/2$ when x is in $[w - L, w]$. Corollary 1 directly deduces a lower bound for $\theta(0, L)$, and Corollary 2 considers the case where $L = w$ and the index of relative risk aversion $R(x)$ is larger or equal to one.[1] In this case, the lower bound of $\theta(0, L)$ is the (non-weighted) average of $A(x)$ when $x \in [0, w]$.

Proposition 1 *For all $L > 0$, we have*

$$\theta(0, L) = \frac{1}{2} \int_{w-L}^{w} [k(x)A(x) \exp\{\int_x^w A(t)dt\}]dx$$

where $k(x) = 2[x - (w - L)]/L^2$ and

$$\int_{w-L}^{w} k(x)dx = 1.$$

Corollary 1 *For all $L > 0$, we have*

$$\theta(0, L) > \frac{1}{2} \int_{w-L}^{w} k(x)A(x)dx.$$

[1] Most empirical studies usually lead to values of $R(x)$ that are larger (and sometimes much larger) than one, and thus the assumption made in Corollary 2 does not seem to be, in practice, very restrictive.

Corollary 2 *If* $L = w$, $R(x) \equiv xA(x) \geq 1$ *for all* x *and* $u(0) \in \mathbb{R}$ *then*

$$\theta(0, L) > \frac{1}{2w} \int_0^w A(x)dx.$$

With the natural case of Decreasing Absolute Risk Aversion (DARA) in mind, Proposition 1 and its corollaries suggest that $\theta(0, L)$ may be large if $A(x)$ is large when x goes to $w - L$.

Symmetrically, Proposition 2 shows that, under non-increasing absolute risk aversion, the normalized risk premium $\theta(p, L)$ may be large when p is close to zero only if $A(w-L)$ is very large, that is, only when the individual's risk tolerance is very small in the accident state.

Proposition 2 *Assume* $R(x) \equiv xA(x) \leq \overline{\gamma}$ *for all* $x \in [w - L, w]$. *Then, under non-increasing absolute risk aversion, we have*

$$\theta(0, L) < \frac{(\overline{\gamma} + 1)A(w - L)}{2},$$

and

$$C(p, L) < pL \left[1 + \frac{(\overline{\gamma} + 1)A(w - L)}{2} L \right].$$

Proposition 2 provides upper bounds for the normalized risk premium $\theta(0, L)$ and for the certainty equivalent $C(p, L)$ when the individual displays non-increasing risk aversion. $\overline{\gamma}$ is an upper bound for the index of relative risk aversion $R(x)$ when x is in the interval $[w - L, w]$. The upper bound of $\theta(0, L)$ is proportional to $A(w - L)$, which is the index of absolute risk aversion in the loss state. Consequently, $C(p, L)$ may be non-negligible when p is very small, say as a proportion of loss L, only if $A(w-L)$ is large. On the contrary, assume $A(w - L) = A(w)$, i.e., the index of absolute risk aversion remains constant in $[w - L, w]$. In that case, we would have $R(x) < R(w)$ for all $x < w$, and thus $\overline{\gamma} = R(w)$, which implies

$$C(p, L) < pL \left[1 + \frac{R(w)}{2} + \frac{R(w)^2}{2} \right].$$

Assuming $R(w) = 2$ or 3 would give $C(p, L) < 4pL$ or $C(p, L) < 7pL$, respectively. Thus, if p is very small, then $C(p, L)/L$ is very small.[2]

[2] For the sake of numerical illustration, consider the case of a large scale nuclear disaster that may occur with probability $p = 10^{-5}$, with total losses of \$100b evenly spread among 1 million inhabitants (think of people living in the neighborhood of the nuclear plant). In the case of an accident, each inhabitant would suffer a loss $L = \$100,000$, with expected loss pL equal to \$1, and certainty equivalent less than \$4 or \$7, which would be negligible, say as a proportion of their annual electricity expenses. Assuming larger but still realistic values of the index of relative risk aversion would not substantially affect this conclusion.

Thus, under non-increasing absolute risk aversion, we may conclude that the risk premium of low-probability high-severity accidents may be non-negligible (and thus that the coverage of such a risk is a relevant issue) if and only if the risk tolerance is very low in such catastrophic cases.

CRRA preferences are an instance of such a case with $T(x) = x/\gamma$, where γ is the index of relative risk aversion. We then have $T(x) \longrightarrow 0$ and $A(x) \longrightarrow \infty$ when $x \longrightarrow 0$. However, *CRRA* preferences are not very satisfactory from a theoretical standpoint, since the utility is not defined when wealth is nil. This corresponds to discontinuous preferences in which any lottery with zero probability for the zero wealth state is preferred to any lottery with a positive probability for this state. If preferences are of the *HARA* type, then risk tolerance is a linear function of wealth, and we may write $T(x) = \eta + x/\gamma$, with $0 < \eta < 1$ and $\gamma > 0$. In such a case, we have $A'(x) < 0, A(0) = 1/\eta$ and $R(x) > 1$. In particular, the individual's absolute risk aversion index is decreasing but upper bounded. A straightforward calculation then gives

$$\frac{1}{2w}\int_0^w A(x)dx = \frac{\gamma}{2w}\ln\left(1+\frac{w}{\gamma\eta}\right),$$

and thus, Corollary 2 shows that for all $M > 0$, we have $\theta(0, L) > M$ if

$$\eta < \frac{w}{\gamma[\exp(2wM/\gamma) - 1]}.$$

The right-hand side of the previous inequality is positive, decreasing in M and increasing in γ. Thus, $\theta(0, L)$ is arbitrarily large if $\eta = T(0)$ and/or $1/\gamma = T'(x)$ are small enough.

3 Insurance demand for catastrophic risks

We now assume that the individual can purchase insurance for a risk (p, L). Insurance contracts specify the indemnity I in the case of an accident, i.e., when the individual suffers a loss L, and the premium $D(p, I)$ to be paid to the insurer depends on the loss probability p and the indemnity I. As an example, the standard insurance pricing model specifies a price proportional to the expected indemnity $D(p, I) = (1+\lambda)pI$, where λ is the loading factor. More generally, we call

$$d(p, I) = \frac{D(p, I) - pI}{pI},$$

the unit cost of insurance with $d(p, I) = \lambda$ in the case of a constant loading. In the case of catastrophic risk however, $d(p, I)$ is strongly affected by the cost of providing the capital that guarantees the solvency of the insurance scheme and it is likely to depend on p and I. We assume non-decreasing marginal costs $D''_{I^2}(p, I) \geq 0$, and $D'_I(p, I) \in (p, 1)$ rules out corner solutions with $I = 0$ or $I = L$. We also assume that the marginal cost of coverage becomes null as p tends to zero, that is

$$\lim_{p \to 0} D'_I(p, I) \equiv D'_I(0, I) = 0,$$

and consistency conditions $D(p, 0) = D(0, I) = 0$.

The policyholder faces the lottery (w_1, w_2), with corresponding probabilities $1 - p$ and p, where w_1 and w_2 denote the wealth in the no-loss and loss states respectively, with

$$w_1 = w - D(p, I) \tag{2}$$

$$w_2 = w - D(p, I) - L + I \tag{3}$$

The full coverage lottery $w_1 = w_2 = w - D(p, L)$ is preferred to the no coverage lottery $(w, w - L)$ if and only if the willingness to pay $C(p, L)$ is higher than the price of full coverage $D(p, L)$, that is

$$C(p, L) \geq D(p, L).$$

When p goes to zero, l'Hôpital's rule allows us to rewritte the previous condition as

$$\frac{C'_p(0, L) - L}{L} = \theta(0, L)L \geq d(0, L),$$

where $d(0, I) = \lim_{p \to 0} d(p, I)$. Hence the following Lemma.

Lemma 1 $\theta(0, L)L \geq d(0, L)$ *is a necessary and sufficient condition for the agent to prefer full insurance to no insurance when p goes to zero.*

Lemma 1 illustrates the importance of the normalized risk premium $\theta(0, L)$ analyzed in the previous section. For insurance to remain attractive despite the vanishingly low probability of accident, the normalized risk premium has to be larger than the unit cost of insurance $d(0, L)$ divided by the loss. A direct consequence of Lemma 1 is that $\theta(0, L)L \geq d(0, L)$ is a sufficient condition for the optimal (partial) insurance cover to remain positive as p goes to zero.[3]

[3] Indeed, if the individual prefers full coverage to no coverage, extending his opportunity set does not make him switch to zero coverage. It is easy to check that the optimal limit cover (denoted I^* below) is positive when $d(0, L) < [u'(w - L) - u'(w)]/u'(w)$ and that this condition is implied by $\theta(0, L)L \geq d(0, L)$.

Combining Corollary 1 and Lemma 1 yields Corollary 3 that provides conditions on relative risk aversion under which insurance remains valuable for low probability events with high severity.

Corollary 3 *Assume $R(x)$ is non decreasing. If $\lim_{x\to 0} R(x) \geq \lim_{L\to w} d(0, L)$, then for L smaller than w but large enough, the individual prefers full insurance to no insurance (and therefore the optimal cover is positive) when p goes to zero.*

Let us now characterize the optimal insurance coverage for a low probability accident. In the (w_1, w_2) plan represented in Figure 1, the set of feasible lotteries is delimited by a concave curve (drawn for $p = 0.1$, $p = 0.25$ and $p = 0.5$) that represents equations (2) and (3), together with the sign condition

$$w_2 - w_1 + L \geq 0, \tag{4}$$

(or $I \geq 0$). For illustrative purpose, Figure 1 displays the case of a simple pricing rule that relates the premium to the expected value and variance of the loss.[4] The optimal lottery maximizes the individual's expected utility

$$(1 - p)u(w_1) + pu(w_2),$$

in this set of feasible lotteries. It is such that the marginal rate of substitution $-dw_2/dw_{1|Eu=ct.} = (1 - p)u'(w_1)/pu'(w_2)$ is equal to the slope (in absolute value) of the feasible lottery locus, that is

$$(1 - p)D'_I(p, I)u'(w_1) = [1 - D'_I(p, I)]pu'(w_2), \tag{5}$$

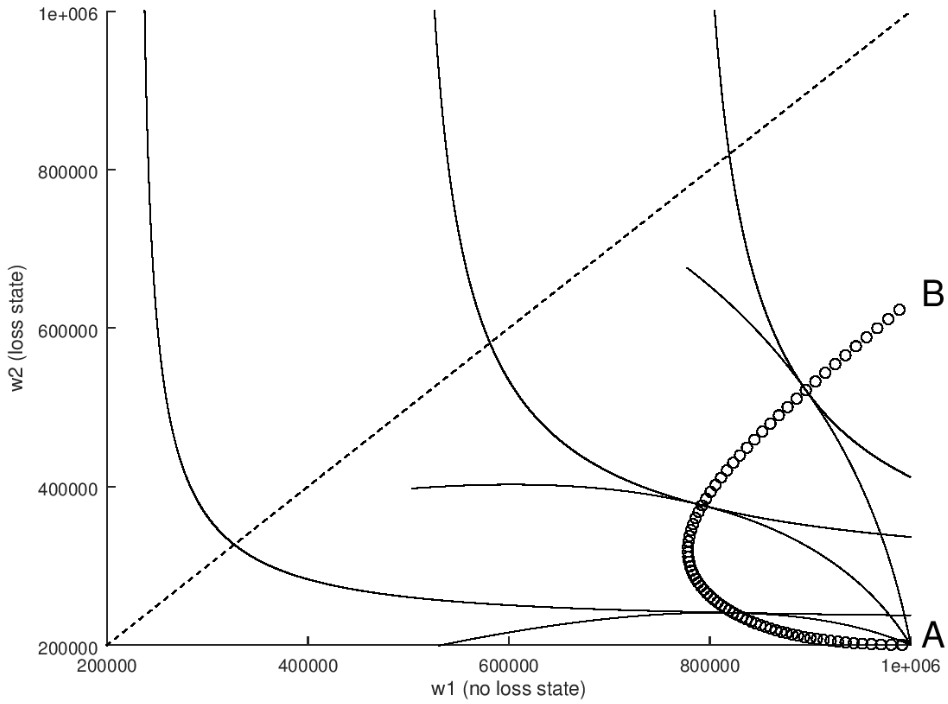
where w_1 and w_2 depend on I through (2) and (3). Figure 1 shows the locus of optimal lotteries in the (w_1, w_2) plane when p changes. Each lottery is at the tangency point of a convex indifference curve with the concave curve that delimits the set of feasible lotteries for a particular probability p. Point A represents the situation with no insurance, and point B represents the asymptotic optimal lottery when p goes to zero.

Let $w_1(p, L)$, $w_2(p, L)$ denote the optimal state-contingent wealth levels when $I > 0$, that is, when $d(p, I)$ is not too large. Equation (5) can be rewritten as

$$\frac{u'(w_2(p, L))}{u'(w_1(p, L))} = \frac{(1 - p)D'_I(p, I)}{p(1 - D'_I(p, I))}.$$

[4]Section 4 will be dedicated to analyzing the supply side of the market and will derive a pricing function.

Figure 1: Comparative statics in the space of lotteries



Each thick black circle represents an optimal lottery for a given probability, from p high in point A to p close to zero in point B. Optimal lotteries are at the tangency point of the indifference curve and the cost curve. In B, the optimal insurance coverage is positive even-though the loss probability is infinitesimally small. The calibration is $w = 1,000,000$, $L = 800,000$, $u(x) = -\frac{x^{-2}}{2}$, and $D(p, I) = (1 + \lambda)pI + \alpha p(1 - p)I^2$ with $\lambda = 0.3$ and $\alpha = 3 \times 10^{-6}$.

Denoting

$$\begin{aligned} w_1^*(L) &\equiv \lim_{p \longrightarrow 0} w_1(p, L) = w \\ w_2^*(L) &\equiv \lim_{p \longrightarrow 0} w_2(p, L), \end{aligned}$$

and using L'Hôpital's rule yields

$$u'(w_2^*(L)) = D''_{Ip}(0, I^*)u'(w_1^*(L)), \tag{6}$$

which implies $w_2^*(L) < w = w_1^*(L)$ if and only if $D''_{Ip}(0, I^*) > 1$. Thus, when p goes to 0, the optimal insurance contract (P, I) goes to a limit (P^*, I^*), with $P^* = D(0, I^*) = 0$ and $I^* = w_2^*(L) + L - w_1^*(L) < L$. When p is positive but close to 0, we still have $I < L$ and $P = D(p, I) \simeq D(p, I^*)$. Since $w_2^*(L) = w - L + I^*$, (6) gives

$$u'(w - L + I^*) = D''_{Ip}(0, I^*)u'(w), \tag{7}$$

which defines I^* when $I^* > 0$, that is when $u'(w - L) > D''_{Ip}(0, 0)u'(w)$ (otherwise $I^* = 0$).

Straightforward calculations allow us to characterize the effect of a change in L and/or w on the asymptotic optimal insurance coverage. An increase $dL > 0$ for w given induces an increase $dI^* < dL$. A simultaneous increase $dw = dL > 0$ induces an increase $dI^* > 0$ in coverage, while an increase in wealth with unchanged loss $dw > 0, dL = 0$ entails a decrease in optimal coverage $dI^* < 0$ under $DARA$ preferences, i.e. when $A' < 0$. Of course, there is nothing astonishing here. These are standard comparative statics results, which are extended to the asymptotic characterization of catastrophic risk optimal insurance. They are summarized in Proposition 3.

Proposition 3 *Assume that $D''_{Ip}(0, 0) \leq \frac{u'(w-L)}{u'(w)}$. Then, when p goes to 0, the optimal insurance coverage I goes to a limit $I^* > 0$ and when p is close to 0, I and P are close to I^* and $D(p, I^*)$, respectively. A simultaneous uniform increase in L and w induces an increase in I and an increase in L induces a lower increase in I^*. Under $DARA$, an increase in w with L unchanged induces a decrease in I and P.*

4 Insurance supply for catastrophic risk

This section determines the pricing rules $D(p, I)$ - taken as given in the previous section - and the corresponding insurance market equilibrium in an economy with a continuum of individuals, complete financial markets and correlated risks.

Assume that a catastrophe occurs with probability π and, in such a case, a fraction $\tilde{\kappa}$ of the population is affected by the loss L (the same for all victims). In order to allow for different severity levels, we assume that $\tilde{\kappa}$ is a random variable, with expected value μ_κ and variance σ_κ^2. For notational convenience, we define

$$\tilde{K} = \begin{cases} \tilde{\kappa} & \text{with probability} \quad \pi \\ 0 & \text{with probability} \quad 1 - \pi \end{cases},$$

the (unconditional) fraction of the population affected by the loss L. Conditionally on the realization of $\tilde{K}$, we assume that all individuals have the same probability to incur the loss L. Therefore, the individual (unconditional) probability of facing a loss L is $p = \pi\mu_\kappa$.

In this setting, the variability of the average loss is associated with a positive coefficient of correlation between individual losses which, as shown in Appendix 7.5, is equal to

$$\rho = \frac{\sigma_\kappa^2 + (1-\pi)\mu_\kappa^2}{\mu_\kappa - \pi\mu_\kappa^2}. \tag{8}$$

The coefficient of correlation ρ increases with σ_κ^2 and since $\tilde{\kappa}$ is distributed over the interval $[0, 1]$, the highest variance is achieved when $\tilde{\kappa}$ is a Bernouilli variable, equal to 0 or 1. The probability that $\tilde{\kappa}$ equals one in this case is simply μ_κ, and we have $\sigma_\kappa^2 = \mu_\kappa(1-\mu_\kappa)$, hence $\rho = 1$. The highest possible variance σ_κ^2 therefore yields perfect correlation across losses.[5]

Another knife-edge case arises when $\sigma_\kappa^2 = 0$ and $\pi = 1$ that is, the same fraction of the population is always affected by the loss. This corresponds to the standard situation where the law of large number applies : the average loss is constant, and the correlation coefficient is $\rho = 0$.

More generally, the correlation coefficient depends on the parameters μ_κ, σ_κ^2 and π and converges to

$$\frac{\sigma_\kappa^2 + \mu_\kappa^2}{\mu_\kappa}, \tag{9}$$

when $\pi \to 0$.

For a given probability distribution of $\tilde{K}$, the insurance provider offers a coverage I at price D to all agents in the economy. For each indemnity paid, the insurer incurs a proportional cost at rate λ because of administrative expenses such as auditing and expertise, or other forms of transaction costs. Furthermore, in order to avoid a default,[6] the insurance provider has to collateralize the random total indemnity costs $\tilde{K}(1+\lambda)I$. This may be achieved through various forms of contracting such as raising equity, in which case the insurance company's equity holders are liable for the policy payments, or by raising capital on financial markets, in which case the risk is transferred to dedicated market investors. Formally, this is equivalent to assuming that the insurers collateralize indemnities by purchasing an asset that delivers a random payoff $\tilde{y} = \tilde{K}(1+\lambda)$, whose price defines D.

[5]When $\pi > 0$, this corresponds to the degenerate case where the catastrophe may be completely harmless. Only two events are then relevant : either there is a catastrophe that affects all individuals, either nobody suffers a loss, hence the perfect correlation.

[6]This paper abstracts from the possibility of default, examined in Charpentier & Le Maux (2014) and Zanjani (2002) for example. In other words, we consider unlimited liability for all agents in the economy and in particular for the insurance provider.

In order to characterize this price, we consider a setting with complete financial markets, where all individuals have the same utility function u. The value of the asset $\tilde{y}$ can be estimated with a standard one factor model written as

$$D = \mathbb{E}\tilde{y} - A(z_0)\text{cov}(\tilde{y}, \tilde{z}). \tag{10}$$

In Equation (10), $\tilde{z}$ is the representative agent's random wealth (defined below) and $A(z_0)$ is his index of absolute risk aversion evaluated at z_0 defined by $u'(z_0) = \mathbb{E}u'(\tilde{z})$. We normalize the utility function by assuming $\mathbb{E}u'(\tilde{z}) = 1$ and thus $u'(z_0) = 1$.[7] Under market completeness, the exposure of investors to idiosyncratic risks have been eliminated from the economy thanks to adequate contracting. As a consequence, equation (10) indicates that only systemic risks give rise to a risk premium in equilibrium. The covariance term of equation (10) captures the idea that investors, who accept to provide the necessary capital to sustain the insurance scheme, require a risk premium to provide a payoff which is high when the average wealth is low.

The representative agent has a wealth equal to the average wealth in the economy:

$$\tilde{z} = w - \tilde{K}L, \tag{11}$$

which is the difference between the exogeneously given level of initial wealth w and the average loss per individual in the economy. A simple calculation using (8), (10) and (11) with $p = \pi\mu_\kappa$ and $\mathbb{E}\tilde{y} = pI$ gives

$$D(p, I) = \psi(p)pI \qquad \forall p \in [0, \mu_\kappa], \tag{12}$$

where

$$\psi(p) = (1 + \lambda)[1 + A(z_0)L(1 - p)\rho], \tag{13}$$

[7] In an economy with complete financial markets, $D = \mathbb{E}\tilde{y} + \text{cov}(\tilde{y}, \frac{u'(\tilde{z})}{\mathbb{E}u'(\tilde{z})})$ defines the price of the asset $\tilde{y}$ (see Gollier (2004)). Using the normalization $\mathbb{E}u'(\tilde{z}) = 1$ and approximating $u'(z)$ in the neighbourhood of z_0 defined by $u'(z_0) = \mathbb{E}u'(\tilde{z})$, gives equation (10). The relationship holds exactly if investors have quadratic utility functions, or if they have CARA utility functions and returns are normally distributed. With any other utility function, it remains a good approximation if the risk $\tilde{y}$ is sufficiently small compared to the market's wealth. Catastrophic risks such as floods, hurricanes, or even nuclear accidents may have systemic consequences that remain small compared to the level of aggregate wealth. As an example, the Japanese government expects the total cost of the Fukushima-Daichi accident in 2011 to amount to a 177 billion euros bill. At least part of this cost is systemic but its size, when compared to a 4300 billion euros annual GDP, remains limited. For larger risks, the exact one factor model could be used without altering our main message.

is the total loading that reflects both the transaction and capital costs of the insurance coverage. Everything else constant, the larger L and/or ρ the larger the insurance price, which reflects the systemic nature of the risk.

Since $p \to 0$ when $\pi \to 0$ with μ_κ given, the condition that guarantees that insurance take-up remains positive in the limit when π tends to 0 can be derived from Proposition 3 as[8]

$$\psi(0) \leq u'(w - L),$$

or equivalently

$$(1 + \lambda)[1 + A(w)L\rho] \leq u'(w - L). \tag{14}$$

In addition, Equation (7) from the previous section also delivers a closed-form solution for the optimal asymptotic coverage

$$I^* = u'^{-1}(\psi(0)) - w + L, \tag{15}$$

where

$$\psi(0) = (1 + \lambda)[1 + A(w)L\rho].$$

An increase in the correlation coefficient ρ reduces the equilibrium coverage I^* since it makes insurance more expensive. An increase in L has a potentially ambiguous effect. Since in our model, policyholders also provide the capital that sustain the insurance scheme, higher losses increase the equilibrium premium. Similarly, an increase in the index of absolute risk aversion is associated with a high demand but also with high premiums.

In order to focus attention on the cost of capital, the following paragraphs restrict attention to the case $\lambda = 0$. When $\rho = 0$, we obtain full coverage in this case, i.e. $I^* = L$, and Proposition 4 with Corollary 4 characterize the asymptotic coverage when $\rho > 0$.

Proposition 4 *If $\lambda = 0$, then, for any utility function u such that $u''' > 0$ and any risk characteristics $(L, \mu_\kappa, \sigma_\kappa^2)$, the asymptotic insurance coverage I^* is positive.*

Corollary 4 *Assume $\lambda = 0$, and let $\beta^*(L) = I^*(L)/L$ be the optimal asymptotic rate of coverage, written as a function of L, everything else given. Then, for any utility function u such that $u''' > 0$ and any risk characteristics $(L, \mu_\kappa, \sigma_\kappa^2)$, we have*

$$\frac{d\beta^*}{dL} > 0.$$

[8]When we consider economies that differ through the value of π, we maintain the normalizing assumption $\mathbb{E}u'(\tilde{z}) = 1$ for the economy. Due to this normalization, the utility function depends on π and $u'(w) \to 1$ when $\pi \to 0$.

Furthermore, β^* is decreasing when ρ is increasing, everything else given and we have

$$\beta^*(L) > 1 - \rho \quad \textit{for all } L.$$

Proposition 4 shows that, when agents are prudent, and in the absence of transaction costs, a systemic risk generates a positive level of coverage in equilibrium. Corollary 4 further shows that the equilibrium rate of coverage depends on the correlation between losses. When ρ is large, the lower bound $1-\rho$ of the coverage rate β^* is low. Hence, in the absence of transaction costs, competitive insurance markets with prudent individuals, backed by complete financial markets, should lead to some coverage of low probability - high severity risks, in spite of the systemic nature of such risks. The larger the loss incurred by victims in the case of a catastrophe, the higher the coverage rate but this coverage rate decreases with the correlation between individual losses.

In contrast with these results that provide a positive view on the ability of insurance markets to cover catastrophic risks, transaction costs and market failures, deliberately ignored here, are likely to play a significant role when the observed coverage of low probability - high severity risks is weak. In particular, limited liability and concerns about default may play an important role as suggested in Zanjani (2002) and Cummins et al. (2002). High transaction costs between insurers and reinsurers also limit the ability of insurers to pay for "the big one" as suggested in Froot (2001) and Niehaus (2002). Imperfect information on risk exposure or biases in beliefs about such risks may also deter individuals from purchasing insurance.

5 Numerical application

This section conducts numerical simulations that illustrate our theoretical results. We assume that policyholders display harmonic absolute risk aversion (HARA) preferences, characterized by

$$u(x) = \zeta\Big(\eta + \frac{x}{\gamma}\Big)^{1-\gamma},$$

whose domain is such that $\eta + (x/\gamma) > 0$, and with the condition $\zeta(1-\gamma)/\gamma > 0$, that guarantees that $u(x)$ is increasing and concave. HARA preferences correspond to affine risk tolerance $T(x) = 1/A(x) = \eta + x/\gamma$, and the coefficient of relative risk aversion is

$$R(x) = x\Big(\eta + \frac{x}{\gamma}\Big)^{-1}. \tag{16}$$

The HARA class nests the constant relative risk aversion (CRRA) case when $\eta = 0$, and the constant absolute risk aversion (CARA) case when $\gamma \to +\infty$. Using the CRRA specification, studies on individual data, such as Levy (1994) and Szpiro (1986), have isolated a plausible range between 1 and 5 for the index of relative risk aversion. We therefore perform simulations over this plausible range of values. In particular, we calibrate the coefficient of relative risk aversion in the no-loss state $R(w) \equiv \overline{R} = 3$ and we let the coefficient of relative aversion in the loss state $R(w-L) \equiv \underline{R}$, vary between 1 and 5. When $\overline{R} = \underline{R}$, the HARA specification boils down the CRRA utility function.

Solving the agent's optimization program in the HARA case and with the pricing rule (12) yields a closed form solution for all acceptable values of p

$$I^{**}(p) = \frac{L + (\eta\gamma + w)(\chi(p) - 1)}{1 + \psi(p)p(\chi(p) - 1)}, \quad \text{where} \quad \chi(p) = \Big[\frac{1 - \psi(p)p}{\psi(p)(1-p)}\Big]^{\frac{1}{\gamma}}, \quad (17)$$

which indeed converges to the value given by (15) when $p \to 0$.

We simulate these optimal coverage levels for a catastrophe of probability $\pi = 1\%$ and $\mu_\kappa = 0.1$. That is, the fraction of the population expected to incur a loss in case of catastrophe is 10 %. The individual loss probability is therefore $p = 0.1\%$. In addition, the variance of $\tilde{\kappa}$ is set at $\sigma^2_{\tilde{\kappa}} = 0.001$, which gives a coefficient of correlation $\rho = 0.109$.[9] We also compute the premium $\psi^{**}(p)$ associated with the optimal levels of coverage and the relative difference

$$\varepsilon(p) = \frac{I^{**}(p) - I^*}{I^{**}(p)},$$

between the true optimal coverage $I^{**}(p)$ and its asymptotic value I^*, characterized by Equation (17) and Proposition 3, respectively.

These simulations are reported in Table 1. The size of the loss L varies across lines from 200,000 euros to 800,000 euros to capture the monetary consequences of a severe accident. Initial wealth w is set at one million euros, which represents roughly the average lifetime discounted earnings of a French citizen. We assume $\overline{R} = 3$ and $\underline{R}$ varies across columns between 1 and 5. Finally, the insurance provider's costs are captured by a proportional loading factor $\lambda = 0.3$.

Optimal coverage increases with the size of the loss and with the coefficient of risk aversion $\overline{R}$. For a given level of risk aversion $\underline{R} = 3$, it rises from 97,010 to 647,899 euros when L increases from 200,000 to 800,000 euros. Fixing the loss at 800,000 euros results in a limited increase in optimal

[9] For simplicity, our calibration uses the first-order approximation $\mathbb{E}(u(\tilde{z})) = u(\mathbb{E}(\tilde{z}))$, which is valid for a small risk (see footnote 9). This allows to calibrate the utility function with the first two moments of the distribution of $\tilde{z}$.

$L \backslash \underline{R}$		1	2	3	4	5
200,000	I^{**}	37,117	85,163	97,010	102,325	105,338
	ψ^{**}	1.38	1.39	1.39	1.39	1.39
	ε	(0.35)	(0.07)	(0.07)	(0.07)	(0.07)
400,000	I^{**}	253,296	273,467	279,373	282,189	283,836
	ψ^{**}	1.47	1.47	1.47	1.47	1.47
	ε	(0.07)	(0.07)	(0.07)	(0.08)	(0.08)
600,000	I^{**}	448,941	459,697	463,056	464,696	465,667
	ψ^{**}	1.56	1.56	1.56	1.56	1.56
	ε	(0.07)	(0.07)	(0.07)	(0.07)	(0.08)
800,000	I^{**}	641,562	646,347	647,899	648,668	649,126
	ψ^{**}	1.64	1.64	1.64	1.64	1.64
	ε	(0.07)	(0.07)	(0.07)	(0.07)	(0.07)

Table 1: Optimal coverage I^{**} in euros, optimal loading ψ^{**}, and relative error ε, at $\pi = 1\%$, $\mu_\kappa = 0.1$ and $\sigma_\kappa^2 = 0.001$ (hence $\rho = 0.109$).

coverage from $641,562$ euros when $\underline{R} = 1$ to $649,126$ euros when $\underline{R} = 5$. The comparative statics with respect to $\underline{R}$ is of particular interest because it summarizes two distinct effects. One the one hand, higher risk aversion pushes toward higher demand levels. On the other hand, it also also yields higher supply prices through the risk premium required in equilibrium. In our simulation, the demand effect dominates the price effect because the risk considered are very large at the individual level but remain limited compared to the overall wealth of the economy.

Coverage rates are limited for lower losses levels, but they increase rapidly with the size of the loss. Because we have fixed relative risk aversion in the loss state $\underline{R} = (w - L)A(w - L)$, the index of absolute risk aversion becomes large as the loss L gets closer to initial wealth w. These higher levels of absolute risk aversion in the loss state explain the higher coverage rates observed for higher loss levels, as we have shown in the previous sections.

The second number of each box represents the total loading ψ^{**} associated with the optimal coverage. For $L = 800,000$ and $\underline{R} = 3$, policyholders pay a premium equal to 1.64 times their expected loss. Since the exogenous loading factor λ was set at 0.3, this implies that the cost of capital is responsible for a 34 percentage points increase in the loading factor. The loading factor increases with both L and $\underline{R}$ but the effect of $\underline{R}$ is quantitatively very limited because $\underline{R}$ only affects the insurance cost through $A(z_0) = R(z_0)/z_0$. As long as the marginal utility certainty equivalent wealth z_0 remains a significant number, a unit change in $R(z_0)$ only leads to limited changes in $A(z_0)$. In contrast, an increase in L directly affect the cost of insurance as Equation (13) shows.

Finally, bracketed numbers report the error in percent, $\varepsilon(p)$, that one would be making by approximating $I^{**}(p)$ with I^*. It varies between 0.35% when $L = 200,000$ euros and $\underline{\text{R}}= 1$ and 0.07% in most scenarios, which is negligible.

These low errors confirm that I^* is an interesting quantity to look at when considering insurance decisions for low probability risks. In addition, large catastrophes often affect people with sometimes widely different (but small) probabilities. Our result suggests that such differences in risk exposure may actually result in very limited differences in optimal coverage values. Similarly, conflicting expert opinions concerning the true probability of a catastrophe would also be irrelevant for the choice of an optimal level of coverage, as long as experts agree that the probability π is very small.

L \ $\underline{R}$		1	2	3	4	5
200,000	I^{**}	20,827	76,844	90,363	96,385	99,785
	ψ^{**}	1.42	1.42	1.42	1.42	1.42
	ε	(0.72)	(0.06)	(0.05)	(0.06)	(0.06)
400,000	I^{**}	235,101	260,051	267,255	270,672	272,665
	ψ^{**}	1.53	1.53	1.53	1.53	1.53
	ε	(0.06)	(0.06)	(0.07)	(0.07)	(0.07)
600,000	I^{**}	428,376	442,128	446,386	448,458	449,683
	ψ^{**}	1.65	1.65	1.65	1.65	1.65
	ε	(0.06)	(0.06)	(0.07)	(0.07)	(0.07)
800,000	I^{**}	619,162	625,395	627,410	628,405	628,999
	ψ^{**}	1.77	1.77	1.77	1.77	1.77
	ε	(0.06)	(0.06)	(0.07)	(0.07)	(0.07)

Table 2: Optimal coverage I^{**} in euros, optimal loading ψ^{**}, and relative error ε, at $\pi = 1\%$, $\mu_\kappa = 0.1$ and $\sigma_\kappa^2 = 0.005$ (hence $\rho = 0.149$).

As a final exercise, Table 2 considers the case of a higher variance $\sigma_\kappa^2 = 0.005$ and therefore of a higher correlation $\rho = 0.149$. The role of the systemic component of the risk is well illustrated here. With a higher variance σ_κ^2, it is more costly to provide insurance, so all loadings are higher. At the same time, the expected probability of loss for a given individual remains constant, hence the lower levels of coverage optimally purchased.

6 Conclusion

The purpose of this paper was to analyze the key determinants of the optimal insurance for catastrophic risks associated with low-probability high-severity events. Considering the limit case of a vanishingly low probability of loss, we have analyzed how insurance demand is affected by the degree of risk aversion when individuals face large scale risks, and the price of insurance. On the supply side, the correlation of losses that characterizes catastrophic risks results in a risk premium in equilibrium. Added to transaction costs, this translates into higher prices for policyholders.

This cost is a well-known impediment to the insurability of catastrophic risks. Our analysis however, demonstrates that this impediment alone hardly explains why some low probability - high severity risks remain uninsured. In particular, in the absence of transaction costs and under complete financial markets, the cost of capital itself should not prevent the insurability of catastrophic risks when individuals are prudent. The larger the loss incurred by victims and/or the lower the correlation between individual losses, the larger the equilibrium rate of insurance coverage. Insurance transaction costs and capital market imperfections, increasing further the cost and lowering the availability of capital therefore play an important role in explaining the failure to insure catastrophic risks. Other market failures, such as asymmetries of information or lack of competition may also contribute to the absence of insurance markets for the more systemic lines of risk. From a normative economics perspective, our analysis also suggests that it is possible and socially desirable for economic policies to organize coverage schemes against catastrophic risks, when transaction and capital costs are offset by the high degree of risk aversion of individuals who face the risk of possible large scale losses.

This may go through policy measures that reduce transaction costs or capital costs for risks that would otherwise be uninsurable. For example, the U.S. National Flood Insurance Program aims to lower insurance transaction costs, by making underwriting and claims handling easier, while also encouraging local communities to take prevention measures that reduce future flood damage. In the field of nuclear risk, international conventions have endorsed a common nuclear corporate liability law, thereby reducing the claims handling costs in the case of a nuclear accident. As for the reduction of capital costs, insurance pools provide examples of what proactive policies can do. This is the case for natural disaster risk (e.g., Flood Re in the UK, or the Caribbean Catastrophe Risk Insurance Facility for hurricanes and earthquakes, to mention just two examples), for large-scale terrorism risk (e.g., the GAREAT pool in France), and for the nuclear risk in almost all countries with nuclear power plants. By contrast, failing to pass onto policyholders the cost of capital due to the systemic component of catastrophic risks would send biased signals to policyholders. The consequences of such an approach on risk prevention behaviors and on the perception of the size of cross-subsidization between more or less exposed individuals are sometimes at the origin of criticisms made towards state-sponsored insurance regimes, such as the natural disaster insurance regimes in France and Spain.

Coming back to more specific results, our numerical analysis only reveals very small differences between the optimal insurance schemes and their asymptotic values, hence validating the relevance of the asymptotic approach. It also suggests that heterogeneity of belief or heterogeneity of exposure to low probability - high severity risks may yield only limited disagreements when it comes to the design of an optimal insurance scheme. This finding is of practical importance for the design of state insurance schemes related to emerging risks such as climate related disasters, large scale terrorism, cyber risks and nuclear accidents.

7 Proofs

7.1 Proof of Proposition 1

From equation (2), we have

$$C_p'(0,L) = \frac{u(w) - u(w-L)}{u'(w)} = \int_{w-L}^{w} \frac{u'(x)}{u'(w)} dx.$$

Since

$$u'(x) = u'(w) - \int_x^w u''(t)dt,$$

for all $x \in [w-L, w]$, we may write

$$\begin{aligned} C_p'(0,L) &= L - \int_{w-L}^{w} \left[\int_x^w \frac{u''(t)}{u'(w)} dt \right] dx \\ &= L + \int_{w-L}^{w} \left[\int_x^w A(t) \frac{u'(t)}{u'(w)} dt \right] dx, \end{aligned}$$

and thus

$$\theta(0,L) = \frac{1}{L^2} \int_{w-L}^{w} \left[\int_x^w A(t) \frac{u'(t)}{u'(w)} dt \right] dx.$$

Integrating by parts gives

$$\theta(0,L) = \frac{1}{2} \int_{w-L}^{w} k(x) A(x) \frac{u'(x)}{u'(w)} dx, \tag{18}$$

where $k(x) = 2[x - (w-L)]/L^2$, with

$$\int_{w-L}^{w} k(x)dx = 1.$$

In addition, we have

$$u'(x) = u'(w) \exp\{\int_x^w A(x)dx\},$$

which completes the proof.

7.2 Proof of Corollary 2

When $L = w$, we have

$$\theta(0,L) > \frac{1}{w} \int_0^w \frac{xu'(x)}{wu'(w)} A(x)dx,$$

from Proposition 1. Furthermore, we have

$$\begin{aligned}\frac{d[xu'(x)]}{dx} &= xu''(x) + u'(x) \\ &= -u'(x)[R(x) - 1],\end{aligned}$$

and thus

$$\frac{d[xu'(x)]}{dx} \leq 0 \text{ if } R(x) \geq 1.$$

We deduce

$$\theta(0, L) > \frac{1}{w}\int_0^w A(x)dx \text{ if } R(x) \geq 1.$$

7.3 Proof of Proposition 2

Using $A' \leq 0$ in equation (18) allows us to write

$$\theta(0, L) \leq \frac{A(w-L)}{L^2 u'(w)} \int_{w-L}^{w} [x - (w-L)]u'(x)dx$$

Using $R(x) \leq \overline{\gamma}$ and $u''(x) < 0$ yields

$$\begin{aligned}\frac{d}{dx}[(x - (w-L))u'(x)] &= u'(x)\Big[1 - R(x) - \frac{u''(x)}{u'(x)}(w-L)\Big] \\ &\geq u'(x)[1 - R(x)] \\ &\geq u'(x)(1-\overline{\gamma}) \\ &\geq u'(w)(1-\overline{\gamma}),\end{aligned}$$

for all $x \in [w-L, w]$. Hence, we have

$$\begin{aligned}\Big[x - (w-L)\Big]u'(x) + (w-x)u'(w)(1-\overline{\gamma}) &\leq [w - (w-L)]u'(w) \\ \Big[x - (w-L)\Big]u'(x) &\leq Lu'(w) + (w-x)u'(w)(\overline{\gamma}-1) \\ &= u'(w)[L + (w-x)(\overline{\gamma}-1)],\end{aligned}$$

for all $x \in [w-L, w]$. Consequently,

$$\begin{aligned}\theta(0, L) &\leq \frac{A(w-L)}{L^2 u'(w)} \int_{w-L}^{w} \{u'(w)[L + (w-x)(\overline{\gamma}-1)]\}\, dx \\ &= \frac{A(w-L)}{L^2}\left[\frac{L^2(\overline{\gamma}+1)}{2}\right] \\ &= \frac{A(w-L)(\overline{\gamma}+1)}{2}.\end{aligned}$$

Using $C_p'' < 0$ and $C(0, L) = 0$ allows us to write

$$\begin{aligned} C(p, L) &< C'(0, L)p \\ &= pL + \theta(0, L)pL^2 \\ &\leq pL\left[1 + \frac{A(w-L)(\overline{\gamma}+1)L}{2}\right]. \end{aligned}$$

7.4 Proof of Corollary 3

Combining Lemma (1) with Corollary (1), shows that

$$d(0, L) \leq \frac{L}{2}\int_{w-L}^{w} \frac{k(x)}{x} R(x)dx$$

is a sufficient condition insurance take-up to be positive.

If $R(x)$ is non decreasing, then

$$\begin{aligned} \frac{L}{2}\int_{w-L}^{w} \frac{k(x)}{x} R(x)dx &\geq \frac{LR(w-L)}{2}\int_{w-L}^{w} \frac{k(x)}{x} dx \\ &= \frac{R(w-L)}{L}\int_{w-L}^{w} \frac{x-(w-L)}{x} dx \\ &= R(w-L)[1-(\frac{w-L}{L})\ln\frac{w}{w-L}] \\ &\equiv \Psi(L) \quad L \in [0, w]. \end{aligned}$$

Noticing that $\lim_{L\to w}\psi(L) = \lim_{x\to 0} R(x)$ provides the result.

7.5 Coefficient of correlation ρ

Let $\tilde{L}_i$ and $\tilde{L}_j$ be two random variables that represent the losses of individuals i and j. Conditionally on a realization κ of the random variable $\tilde{\kappa}$, losses are assumed identically and independently distributed, hence

$$\tilde{L}_i\tilde{L}_j|\kappa = \left\{ \begin{array}{lll} L^2 & \text{with probability} & \pi\kappa^2 \\ 0 & \text{with probability} & 1-\pi\kappa^2 \end{array} \right. .$$

As a consequence $\mathbb{E}(\tilde{L}_i\tilde{L}_j|\kappa) = L^2\pi\kappa^2$ and $\mathbb{E}(\tilde{L}_i\tilde{L}_j) = L^2\pi\mathbb{E}\tilde{\kappa}^2$. Similarly,

$$\tilde{L}_i|\kappa = \left\{ \begin{array}{lll} L & \text{with probability} & \pi\kappa \\ 0 & \text{with probability} & 1-\pi\kappa \end{array} \right. ,$$

for all i, hence $\mathbb{E}\tilde{L}_i = L\pi\mu_\kappa$ and $\mathbb{E}\tilde{L}_i\mathbb{E}\tilde{L}_j = (L\pi\mu_\kappa)^2$. The co-variance between two losses is therefore written as

$$\text{cov}(\tilde{L}_i, \tilde{L}_j) = L^2\pi[\mathbb{E}\tilde{\kappa}^2 - \pi\mu_\kappa^2].$$

Also, since

$$\tilde{L_i}^2 | \kappa = \begin{cases} L^2 & \text{with probability} \quad \pi\kappa \\ 0 & \text{with probability} \quad 1-\pi\kappa \end{cases},$$

implies $\mathbb{E}(\tilde{L_i}^2) = \pi L^2 \mu_\kappa$, we find the variance of $\tilde{L}_i$

$$\mathrm{V}(\tilde{L}_i) = L^2 \pi \mu_\kappa (1 - \pi\mu_\kappa).$$

Since $\mathrm{V}(\tilde{L}_i) = \mathrm{V}(\tilde{L}_j)$ for all i, the coefficient of correlation is finally equal to

$$\begin{aligned} \rho &= \frac{\mathrm{cov}(\tilde{L}_i, \tilde{L}_j)}{\mathrm{V}(\tilde{L}_i)} \\ &= \frac{\mathbb{E}\tilde{\kappa}^2 - \pi\mu_\kappa^2}{\mu_\kappa(1-\pi\mu_\kappa)} \\ &= \frac{\sigma_\kappa^2 + \mu_\kappa^2(1-\pi)}{\mu_\kappa(1-\pi\mu_\kappa)}, \end{aligned}$$

where the last line is obtained using $\sigma_\kappa^2 = \mathbb{E}\tilde{\kappa}^2 - \mu_\kappa^2$.

7.6 Proof of Proposition 4

With $\lambda = 0$, inequality (14) is rewritten as

$$1 + A(z_0) L \rho \leq u'(w - L). \tag{19}$$

Making use of the normalization $\mathbb{E}(u'(\tilde{z})) = 1$,[10] then yields

$$\begin{aligned} -u''(w) L \rho &\leq u'(w-L) - u'(w) \\ &= -\int_{w-L}^{w} u''(x) dx. \end{aligned} \tag{20}$$

If $u'''(x) > 0$, we have $u''(x) < u''(w)\ \forall x \in [w - L, w]$. So

$$\int_{w-L}^{w} u''(x) dx < u''(w) L$$

For (20) to hold, it is therefore sufficient that

$$-u''(w) L \rho \leq -u''(w) L,$$

which is always true since $\rho \leq 1$ and $u'' < 0$.

[10]See footnote 8.

7.7 Proof of Corollary 4

Equation (15) with $\lambda = 0$ gives

$$\beta^*(L) = 1 + \frac{u'^{-1}(1 + A(w)L\rho) - w}{L}, \tag{21}$$

from which we derive

$$\frac{d\beta^*}{dL} = \frac{A(w)L\rho/u''(u'^{-1}(1 + A(w)L\rho)) - [u'^{-1}(1 + A(w)L\rho) - w]}{L^2}.$$

$d\beta^*/dL$ is therefore positive if

$$\frac{A(w)L\rho}{-u''(u'^{-1}(1 + A(w)L\rho))} < w - u'^{-1}(1 + A(w)L\rho). \tag{22}$$

Also, $u''' > 0$ and $u'' < 0$ implies that u'^{-1} is decreasing convex. Therefore

$$u'^{-1}(1 + A(w)L\rho) \quad < \quad u'^{-1}(1) + \frac{A(w)L\rho}{u''(u'^{-1}(1 + A(w)L\rho)}$$

For (22) to hold, it is sufficient that

$$\frac{A(w)L\rho}{-u''(u'^{-1}(1 + A(w)L\rho))} \leq w - u'^{-1}(1) + \frac{A(w)L\rho}{-u''(u'^{-1}(1 + A(w)L\rho)}, \tag{23}$$

which is true since our normalization changes with π in such a way that $u'(w) \to 1$ when $\pi \to 0$. Finally, since u'^{-1} is decreasing, (21) shows that $\beta^*(L)$ is decreasing in ρ. Furthermore, using L'Hôpital rule yields:

$$\begin{aligned}\lim_{L\to 0} \beta^*(L) &= 1 + \lim_{L\to 0} \frac{A(w)\rho}{u''(u'^{-1}(1))} \\ &= 1 - \rho.\end{aligned}$$

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