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Feedback control of social distancing for COVID-19 via elementary formulae

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Abstract.— Social distancing has been enacted in order to mitigate the spread of COVID-19. Like many authors, we adopt the classic epidemic SIR model, where the infection rate is the control variable. Its differential flatness property yields elementary closed-form formulae for open-loop social distancing scenarios, where, for instance, the increase of the number of uninfected people may be taken into account. Those formulae might therefore be useful to decision makers. A feedback loop stemming from model-free control leads to a remarkable robustness with respect to severe uncertainties and mismatches. Although an identification procedure is presented, a good knowledge of the recovery rate is not necessary for our control strategy.

Keywords: Biomedical control, COVID-19, social distancing, SIR model, flatness-based control, model-free control, robustness, identifiability, algebraic differentiator

1. INTRODUCTION

In two years an abundant mathematically oriented literature has been devoted to the worldwide COVID-19 pandemic. Some of the corresponding calculations had even a significant political impact (see, e.g., Adam (2020); Quintana et al. (2021)). Note that in the field of mathematical epidemiology of infectious diseases the role of modeling human behavior became increasingly important in the last 15 years. It gave birth to a novel research field named behavioral epidemiology of infectious diseases: see, e.g., Manfredi & d’Onofrio (2013); Wang et al. (2016).

A novel control technique for improving the social distancing is presented here. This fundamental topic has already been tackled by many authors: see, e.g., Al-Radhawi et al. (2022); Ames et al. (2020); Angulo et al. (2021); Berger (2022); Bisiacco & Pillonetto (2021); Bliman & Duprez (2021); Bliman et al. (2021); Bonmass & Gianatti (2020); Borri et al. (2021); Charpentier et al. (2020); Di Lauro et al. (2021a,b); Dias et al. (2022); Efimov & Ushirobira (2021); Gevertz et al. (2021); Godera et al. (2021); Greene & Sontag (2021); Ianni & Rossi (2021); Jing et al. (2021); Köhler et al. (2021); McQuade et al. (2021); Morato et al. (2020a,b); Morgan et al. (2021); Morris et al. (2021); O’Sullivan et al. (2020); Péni et al. (2020); Pillonetto et al. (2021); Sadeghi et al. (2021); Sontag (2021); Stella et al. (2022); Tsay et al. (2020). Most of those papers are based on the famous SIR (Susceptible-Infected-Recovered/Removed) model, which goes back to 1927 (Kermack & McKendrick (1927)), or on some modifications of its compartments. This communication is also using the SIR model:

• When, like in several papers, the infection rate is the control variable, the SIR model is (differentially) flat (Fliess et al. (1995)). Remember that flatness-based control is one of the most popular model-based control setting, especially with respect to concrete applications: see, e.g., Beltran-Carabajal et al. (2021); Bonnabel & Clays (2020); Diwold et al. (2022); Kogler et al. (2022); Li et al. (2021); Lorenz-Meyer et al. (2020); Miuske (2020); Richter et al. (2021); Sahoo & Chidlarwar (2020); Sanchez et al. (2020); Schörghuber et al. (2020); Stockler et al. (2021); Sekiguchi et al. (2021); Tal & Karaman (2021); Thunthong et al. (2021); Tognon & Franchi (2021); Zauner et al. (2021) for some recent publications. Note that flatness has already been utilized by Hametner et al. (2021) for studying COVID-19 but with other purposes.

• There are severe uncertainties: model mismatch, poorly known initial conditions, ... We therefore close the loop around the reference trajectory via model-free control, or MFC, in the sense of Fliess & Join (2013, 2021a). MFC, which is easy to implement, has already been illustrated in a number of practical situations. Some new contributions are listed here: Gu et al. (2021); Ismail et al. (2021); Jin et al. (2021); Kuruganti et al. (2021); Lv et al. (2022); Manzoni & Rampazzo (2021); Mao et al. (2021); Michel et al. (2022); Mousavi et al. (2021); das Neves & Angelico
(2021); Sancak et al. (2021); Srour et al. (2021); Sun et al. (2021); Xu et al. (2020, 2021); Wang et al. (2022, 2020a); Wang & Wang (2020b); Zhang et al. (2020, 2021); Zhou et al. (2021). Let us single out here the excellent work by Truong et al. (2021) on ventilators, which are related to COVID-19.

In order to be more specific consider a flat system with a single input u and a single output y. Assume that y is a flat output. Our strategy (see also Villagra & Herrera-Pérez (2012); Fliess et al. (2021c)) may be summarized as follows:

1. To any output reference trajectory y* corresponds at once thanks to flatness an open-loop control u*.
2. Let z be some measured output. Write z* the corresponding reference trajectory. Set u = u* + ∆u, where ∆u is the control of an ultra-local local model (Fliess & Join (2013)). Its output ∆z = z − z* is the tracking error. Closing the loop via an intelligent controller (Fliess & Join (2013)) permits to ensure local stability around z* in spite of severe mismatches and disturbances.

Our paper (see Fliess & Join (2021b) for a first draft) is organized as follows.

- Section 2 shows that the SIR model, where the infection rate is the control variable, is flat and the population of recovered/removed individuals is a flat output; the recovery rate is identifiable in the sense of Fliess et al. (2008).
- Section 3 is devoted to a flatness-based control strategy, i.e., to a feedforward approach. Elementary closed-form of the control and state variables are easily derived. Various scenarios, where for instance the number of infected persons is increased, may thus be easily suggested to decision makers.
- Closing the loop via an intelligent proportional regulator, stemming from model-free control, is the subject of Section 4. Computer simulations confirm an excellent robustness with respect to severe uncertainties.
- A time-varying recovery rate is estimated in Section 5 via algebraic estimation methods (Fließ et al. (2008)). Techniques from Section 4 show however good performances if this rate is wrongly assumed to be constant.
- Some suggestions for future investigations and some concluding remarks may be found in Section 6.

2. MODELING ISSUES

2.1 The SIR model

The SIR model (see, e.g., Weiss (2013) for a nice introduction) reads:

\[
\begin{align*}
\dot{S} &= -\beta IS \\
\dot{I} &= \beta IS - \gamma I \\
\dot{R} &= \gamma I
\end{align*}
\]

(1) S, I and R, which are non-negative quantities, correspond respectively to the fractions of susceptible, infected and recovered/removed individuals in the population. We may set therefore

\[
S + I + R = 1
\]

(2) \(\beta, 0 < \beta < \beta \leq \gamma\), which is here the control variable, \(^1\) and the parameter \(\gamma > 0\) are respectively the infection and recovery rates.

2.2 Flatness

Equations (1)-(2) show that System (1) is flat and that \(R\) is a flat output (Fliess et al. (1995)). The other system variables may be expressed as differential rational functions of \(R\), i.e., as rational functions of \(R\) and its derivatives up to some finite order:

\[
\begin{align*}
I &= \frac{\dot{R}}{\gamma} \\
S &= 1 - R - \frac{\dot{R}}{\gamma} \\
\beta &= -\frac{\dot{S}}{TS} = \frac{1}{S} \left( \frac{i}{I} + \gamma \right)
\end{align*}
\]

(3)-(4)-(5)

Remark 1. If \(\gamma\) is not constant, but a differentiable function of time, Equations (3)-(4)-(5) remain valid; System (1) is still flat and \(R\) is still a flat output. Equation (5) shows however that \(\dot{\gamma}\) is needed.

2.3 An addendum on the SEIR model

The SEIR model (see, e.g., Brauer & Castillo-Chavez (2012)) is a rather popular extension of the SIR model:

\[
\begin{align*}
\dot{S} &= -\beta IS \\
\dot{E} &= \beta IS - \alpha E \\
\dot{I} &= \alpha E - \gamma I \\
\dot{R} &= \gamma I
\end{align*}
\]

(6)

where \(\alpha > 0\) is an additional parameter. Equation (2) becomes

\[
S + E + I + R = 1
\]

(7)

Equations (6)-(7) show that the SEIR model is also flat and that \(R\) is a flat output:

\[
\begin{align*}
I &= \frac{\dot{R}}{\gamma} \\
E &= \frac{\dot{I} + \gamma I}{\alpha} = \frac{\dot{R} + \gamma \dot{R}}{\gamma \alpha} \\
S &= 1 - R - I - E = 1 - R - \frac{\dot{R}}{\gamma} - \frac{\dot{R} + \gamma \dot{R}}{\gamma \alpha}
\end{align*}
\]

(8)

2.4 Identifiability of the recovery rate

Equation (5) yields

\[
\gamma = \beta S - \frac{\dot{I}}{I}
\]

\(\gamma\) is a differential rational function of \(R\) and \(\beta\): It is thus rationally identifiable (Fliess et al. (2008)).

Remark 2. The above equation does not work for an identifiability purpose if \(\gamma\) is time-varying: \(\dot{\gamma}\) is sitting

\(^1\) Softening social distancing implies increasing \(\beta(t)\).
on its right hand-side. If we assume that \(I\) and \(S\) are measured, Equation (4) yields
\[
\gamma = \frac{i - \beta IS}{I}
\] (8)
\(\gamma\) is still rationally identifiable with respect to \(I, S, \beta\). It will be useful in Section 5.

3. FLATNESS-BASED CONTROL

3.1 Preparatory calculations

Set
\[
I_{\text{reference}}(t) = I_0 e^{-\lambda t}
\]
where \(t \geq 0, 0 \leq I_0 \leq 1\), and \(\lambda \geq 0\) is some constant parameter. If we set \(R(0) = 0\), it yields
\[
R_{\text{reference}}(t) = \frac{\gamma I_0}{\lambda} \left(1 - e^{-\lambda t}\right)
\]
\[
S_{\text{reference}}(t) = 1 - \frac{\gamma I_0}{\lambda} \left(1 - e^{-\lambda t}\right) - I_0 e^{-\lambda t}
\]
and the open-loop control
\[
\beta_{\text{flat}}(t) = \frac{\gamma - \lambda}{1 - \frac{2I_0}{\lambda} \left(1 - e^{-\lambda t}\right) - I_0 e^{-\lambda t}}
\]

Thus
\[
\lim_{t \to +\infty} \beta_{\text{flat}}(t) = \frac{\lambda(\gamma - \lambda)}{\lambda - \gamma I_0}
\] (9)
The following inequalities are straightforward:
\[
\gamma I_0 < \lambda < \gamma
\] (10)
\(\lambda < \gamma\) follows from \(\beta > 0\); \(\gamma I_0 < \lambda\) follows from
\[
\lim_{t \to +\infty} S(t) = 1 - \frac{\gamma I_0}{\lambda} = S(\infty) > 0
\] (11)
Introduce the more or less precise quantity \(\beta_{\text{accept}}\), where \(\beta < \beta_{\text{accept}} < \beta\). It stands for the “hardest” social distancing protocols which are “acceptable” in the long run. Equation (9) yields therefore
\[
\frac{\lambda(\gamma - \lambda)}{\lambda - \gamma I_0} = \beta_{\text{accept}}
\]
The positive root of the corresponding quadratic algebraic equation \(\lambda^2 + (\beta_{\text{accept}} - \gamma)\lambda - \gamma I_0\beta_{\text{accept}} = 0\) is
\[
\lambda_{\text{accept}} = \frac{\gamma - \beta_{\text{accept}} + \sqrt{\Delta_{\text{accept}}}}{2}
\]
where \(\Delta_{\text{accept}} = (\gamma - \beta_{\text{accept}})^2 + 4\gamma I_0\beta_{\text{accept}} \geq 0\).
The fundamental inequality
\[
\gamma I_0 < \lambda_{\text{accept}} < \gamma
\]
follows from
\[
\lim_{\lambda \gamma I_0} \frac{\lambda(\gamma - \lambda)}{\lambda - \gamma I_0} = +\infty, \quad \lim_{\lambda I_0} \frac{\lambda(\gamma - \lambda)}{\lambda - \gamma I_0} = 0
\]
Equation (11) leads to the notation
\[
S_{\text{accept}}(\infty) = 1 - \frac{\gamma I_0}{\lambda_{\text{accept}}}
\]
The inequality
\[
S(\infty) < S_{\text{accept}}(\infty) \quad \text{if} \quad \lambda < \lambda_{\text{accept}}
\]
demonstrates that the proportion of uninfected people decreases if the social distancing obligations are relaxed.

3.2 Two computer experiments

Set \(\gamma = 0.1, \beta_{\text{accept}} = 0.22\). Figure 1 displays the open-loop evolutions of \(\beta, I, S\) when \(\lambda = \lambda_{\text{accept}}\). Those behaviors are quite satisfactory.

4. MODEL-FREE CONTROL

4.1 Ultra-local model

Set \(\Delta I(t) = I(t) - I_{\text{reference}}(t), \beta(t) = \beta_{\text{flat}}(t) + \Delta \beta(t)\). In order to take into account the various uncertainties, introduce the ultra-local model (Fliess & Join (2013))
\[
\frac{d}{dt} \Delta I = F + a \Delta \beta
\] (12)
- The function \(F\), which is data-driven, subsumes the poorly known structures and disturbances.
- The parameter \(a\), which does not need to be precisely determined, is chosen such that the three terms in Equation (12) are of the same magnitude.

\[
F_{\text{est}} = -\frac{a}{\sigma} \int_{t-\tau}^{t} \left(\Delta I(\sigma) + \sigma(\tau - \sigma) \Delta \beta(\sigma)\right)d\sigma, \quad \tau > 0 \text{ is “small”, gives a real-time estimate, which in practice is implemented via a digital filter.}
\]

4.2 Intelligent proportional controller

Introduce (Fliess & Join (2013)) the intelligent proportional controller, or iP,
\[
\Delta \beta = \frac{-F_{\text{est}} + K_P \Delta I}{a}
\] (13)
where \(K_P\) is a tuning gain. Equations (12) and (13) yield
\[
\frac{d}{dt} \Delta I + K_P \Delta I = F - F_{\text{est}}
\]
Set \(K_P > 0\). Then \(\lim_{t \to +\infty} \Delta I(t) \approx 0\) if the estimate \(F_{\text{est}}\) is “good,” i.e., if \(F - F_{\text{est}}\) is “small.” Local stability is ensured.

Remark 3. When compared to classic PIs and PIDs (see, e.g., Åström & Murray (2008)), the gain tuning of the iP is straightforward.

4.3 Computer experiments

The sampling time interval is 2 hours. In Equations (12) and (13), \(a = 0.1, K_P = 1\). Figure 2 displays excellent results in spite of errors on initial conditions and of the fuzzy character of any measurement of the social distancing. This fuzziness is expressed here by an additive corrupting white Gaussian noise \(\mathcal{N}(0, 5.10^{-3})\) on \(\beta\).

5. ON THE RECOVERY RATE \(\gamma\)

Assume now that \(\gamma\) is a differentiable time function. Equation (8) yields the algebraic estimator
\[
\gamma_{\text{est}} = \frac{[\hat{I}]_{\text{est}} - \beta IS}{I}
\] (14)
where \([\hat{I}]_{\text{est}}\) is an estimate of \(\hat{I}\) obtained along the lines developed by Mboup et al. (2009) and Othmane et al. (2021) for algebraic differentiators. Figure 3-c displays
excellent results. The flatness-based computer experiments is achieved as in Section 3.2, i.e., \( \gamma = 0.1 \) is assumed to be constant. Lack of space prevents us from examining more realistic situations. Closing the loop via model-free control yields as demonstrated in Figures 3-a-b a satisfactory behavior. Is the exact knowledge of the recovery rate unimportant?

6. CONCLUSION

Casella (2021) questions the relevance and usefulness of such control-theoretic considerations for non-pharmaceutical mitigation policies against COVID-19. We certainly do not claim to set aside those objections in this preliminary short study. The combination however of flatness-based and model-free controls presents nevertheless some major advantages as demonstrated here and by Villagren & Herrero-P{é}rez (2012) and Flies et al. (2021c).

An extra theoretical effort must be made in order to design control strategy as close as possible to the real epidemic control enacted by Public Health authorities. Summarizing, we consider this results proposed in this work as a theoretical ideal framework, to be filled with a more realistic picture: an implementable non-pharmaceutical control strategy. Preliminary results, which we recently obtained, indicate that the methodology here proposed is in the right direction (see Join et al. (2022)).

REFERENCES


Figure 1. Open loop control strategy. Trajectories corresponding to two distinct initial conditions for the infectius $I_0 = 0.05$ (single-dashed curves: - - ) and $I_0 = 0.1$ (double-dashed curves: - - -). Left panel: plot of the transition rate $\beta(t)$; central panel: plot of the infectious fraction $I(t)$; right panel: plot of the fraction of susceptible subjects $S(t)$.

Figure 2. Effect of both errors on initial conditions and of the fuzziness of measurements of social distancing. In all panels, dashed blue line represent the reference trajectories. Left panel: plot of the transition rate $\beta(t)$; central panel: plot of the infectious fraction $I(t)$; right panel: plot of the fraction of susceptible subjects $S(t)$.

Figure 3. Impact of the estimation of the time-varying recovery rate $\gamma$. 

(d) $\gamma$ (- - ) and $\gamma_{est}$ (blue - - )